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SJOERD D. ZWART

REFINED VERISIMILITUDE

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STUDIES IN EPISTEMOLOGY,

LOGIC, METHODOLOGY, AND PHILOSOPHY OF SCIENCE

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To Estelle and Laetitia

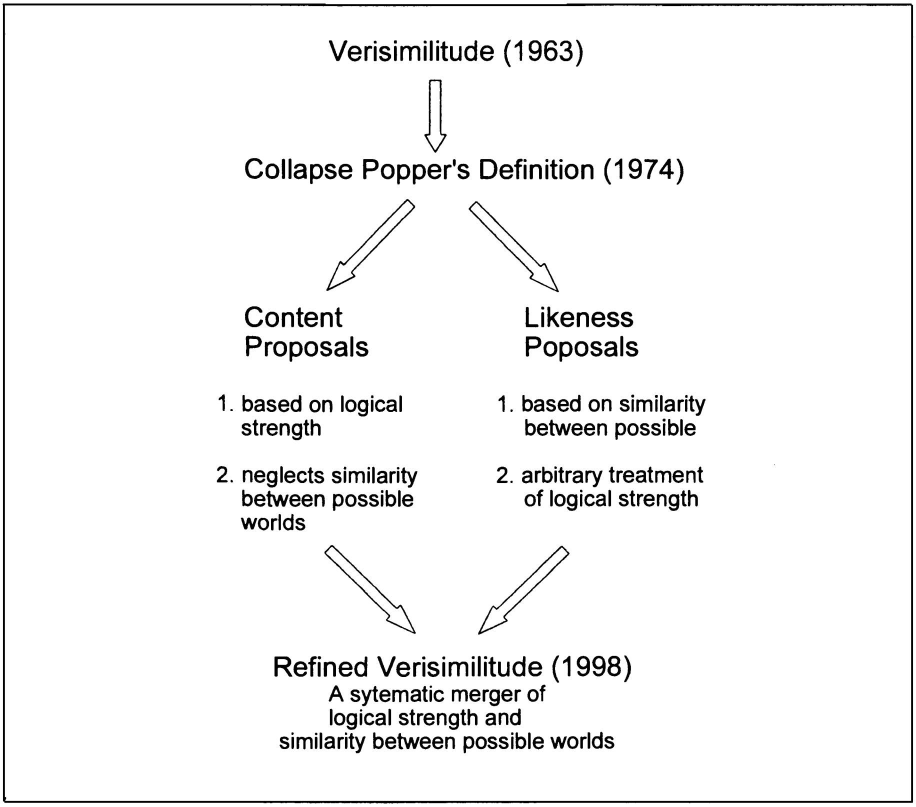
CONTENTS

PREFACE

The subject of the present inquiry is the approach-to-the-truth research, which started with the publication of Sir Karl Popper's *Conjectures and Refutations*. In the decade before this publication, Popper fiercely attacked the ideas of Rudolf Carnap about confirmation and induction; and ten years later, in the famous tenth chapter of *Conjectures* he introduced his own ideas about scientific progress and verisimilitude (*cf.* the quotation on page 6). Abhorring inductivism for its appreciation of logical weakness rather than strength, Popper tried to show that fallibilism could serve the purpose of approach to the truth. To substantiate this idea he formalized the common sense intuition about preferences, that is: B is to be preferred to A if B has *more advantages* and *fewer drawbacks* than A.

In 1974, however, David Miller and Pavel Tichý proved that Popper's formal explication could not be used to compare false theories. Subsequently, many researchers proposed alternatives or tried to improve Popper's original definition. One of my results shows that Oddie is right when he claims that all these alternatives are either *content* or *likeness* proposals. The first base their ordering on truth-value and logical content, and neglect similarity among possible worlds; the second build their ordering primarily on similarity among models (or constituents), and treat logical strength rather arbitrarily (my formal characterization of the difference between likeness and content definitions can be found in Subsection 1.4.3). I compared the way eight alternative definitions order propositions of a finite propositional language, since this shows the barest outline of those alternatives; additionally, I compared their metatheoretical properties. The outcome clearly underlines the difference between likeness and content definitions. For instance, according to the first, the negation of the truth, which is almost a tautology, is the worst proposition; the second claims that all propositions improve the *complete falsehood*, which is the strongest description of the worst possible world.

The current study has the following outline. In Chapter 1, I introduce the differences between verisimilitude and truthlikeness definitions, which are the subjects of Chapters 2–3. Together, the first three chapters form the expository part of the present publication. In the Chapter 4, I formulate and examine the rules of theory-choice that accompany Niiniluoto's and Kuipers's definitions. Chapter 5 concerns the fact that preference relations based on similarity among possible worlds are not invariant under extensional substitutions. This property has misleadingly been called the "language dependency" of truthlikeness definitions. Sketching the solution to this "problem", I prepare the way for my most important contribution to the truth approximation debate, viz. the *refined verisimilitude* definition, which is presented in Chapter 6. In a way, Chapters 1–5 can be viewed as preparatory steps leading up to the presentation of my new preference ordering of propositions in Chapter 6. The predecessors of my proposal are shown in the diagram below.



Verisimilitude and Truthlikeness

It is important to note that the present inquiry does not exclusively concern formal philosophy of science. The question of ordering propositions has a much wider scope, and is of general logical interest. Readers who are mainly concerned with preferences among possible worlds or propositional constituents may skip the Chapters 2–5; and having used Chapter 1 as an intuitive introduction, they may start reading Chapter 6.

Thanks are due to many people with whom I discussed parts of the present publication. First of all I am indebted to Theo Kuipers and Johan van Benthem; the first for coming up with the idea to write on verisimilitude and truthlikeness; the second for being instrumental in finishing the project. Johan van Benthem inspired me to put the approach-to-the-truth project in a much broader context. Further, I owe a special debt to Veikko Rantala and David Miller who both spent time and energy on my solution of the language dependency problem (see Chapter 5). Regrettable, I did not manage to make David Miller change his mind on his constraint of invariance of likeness orderings under extensional substitutions. I owe much to Erik Krabbe, who with remarkable and uncompromising dedication went meticulously through the entire manuscript indicating the places that needed correction or more explanation. It is Krabbe's contribution that increased the readability of the technical details considerably. Thanks are also due to Miranda Aldam-Breary who spent many hours reading the text with a keen 'eye' to the English. It is needless to say that all errors and imprecisions in the present publication remain entirely mine.

Finally, I gratefully acknowledge my stipend as a Grotius Fellow at the University of Amsterdam which enabled me to finish the present publication and to continue my research into the relation between Belief Revision and Verisimilitude. I dedicate the present publication to my beloved wife and daughter.

SJOERD D. ZWART

CHAPTER 1

INTRODUCTION AND TERMINOLOGY

The approach-to-the-truth project started with the publication of Popper's *Conjectures and Refutations* in 1963, which contains the first formal explication of the "verisimilitude" notion. Eleven years after this publication, the twenty-fifth volume of the *British Journal for the Philosophy of Science* gave the project a significant incentive. In their contributions, Miller, Tichý, and Harris, proved the inadequacy of Popper's definition. In this chapter the rise and fall of Popper's proposal is sketched, and a technical framework is developed to compare the alternatives to Popper's proposal. I deal with the general philosophical background of the verisimilitude notion, Popper's definition and its failure in Sections 1.1–1.2 . An explanation of how I shall compare the various proposals presented in Chapters 2–3 is given in Section 1.3. I shall base this comparison on the most elementary mathematical applications of the definitions: propositional languages. The two different ways to paraphrase theories and data in the algebra foreshadow the paramount distinction between two kinds of approach-to-the-truth proposals: the content and likeness definitions. This distinction is reconstructed in Section 1.4 in terms of two different strategies to revise Popper's original explication. It leads to a formal definition of the contrast between content and likeness definitions. Finally, in the fifth section, I introduce further metatheoretical properties used in later chapters.

1.1. PHILOSOPHICAL BACKGROUND

The basic philosophical ideas related to the verisimilitude notion are sketched in this section. My ambition is a modest one. I do not want to give a thorough account of all the advantages and drawbacks of the various points of view involved. That would require a separate publication of more than one volume. It is my intention to give a rough map as to where to find the verisimilitude notion in the landscape of philosophical ideas about developments in scientific knowledge.

1.1.1. Scientific Progress

I want to mention two prescientific or philosophical intuitions underlying the idea of *scientific progress*. The first is that knowledge in various scientific disciplines becomes increasingly exact. I shall call this feature *growth in depth.* Scientists manage to answer more and more why questions about a particular area of investigation. For instance, scientists became acquainted with chemical reactions that caused phenomenological changes, next, they discovered the importance of chemical bonds in chemical substances, and today, chemists use quantum mechanical insights to study these bonds. In the same manner, scientists first thought that atoms where the smallest particles in the universe; subsequently they discovered that atoms consisted of protons, neutrons and electrons, and today, quarks are thought to be the smallest building blocks of all substances. Smaller steps, such as concretizations also exemplify this growth in depth. The Van der Waals's refinement of the Boyle-Gay Lusac law serves as an example. These are not unique examples of the growth in depth phenomenon. On the contrary, the scientific community has produced such examples in large numbers. C.S. Peirce compared the improvement of our scientific understanding with the increasing knowledge about the numerical value of π in digits.[1] Just as our knowledge about π's decimals increases, our knowledge of physical phenomena becomes more and more exact. The idea that this increase of exactness must be interpreted as *accumulation* has been severely scrutinized in the past, since no scientific theory is strictly speaking true.[2] Scientific progress interpreted as approaching the truth differs from progress as accumulation of truths.

The second intuition related to scientific progress is the idea that the *number of scientific subjects,* about which scientists make warranted claims, *increases steadily*: I shall call this phenomenon *growth in breadth.* Not long ago, people believed that earthquakes, the plague, lightning, and deformed babies were all examples of divine punishments. Now we know that plate-tectonics, the bacterium Yersinia pestis, electromagnetism, and DNA provide scientific explanations for these phenomena. Our knowledge about the different subjects is also becoming mutually compatible.[3] This fosters the idea that our knowledge is not an arbitrary way to put phenomena together, but indeed reflects some relevant part of reality. In short, the growth in number of answered questions and the profound increase of detail nurture our philosophical intuitions about improvement of scientific knowledge and advancement that scientists make in unravelling the mysteries of nature. Combining content and a likeness strategies, my proposal of *Refined Verisimilitude* mirrors both forms of progress.

In our era, after the publications of Thomas Kuhn, the distinction between normal science and scientific revolutions has gained importance. The growth in breadth and depth are not incompatible with the distinction between normal and revolutionary science. The ideas about paradigmatic changes only qualify the notion of constant scientific growth. The distinction between normal science and scientific revolutions pertains to the two forms of scientific progress. For instance, the enhancement our specific knowledge about the chemical properties of chemical compounds (*growth in depth*) is a paradigmatic example of normal science; however, it might lead to unexpected results that can end in revolutionary insights.

The debate between realists and instrumentalists, also affects the discussion about scientific progress. The success of science has been used as a weapon against the antirealists: witness the following, often cited quotation of Putnam: "The positive argument for realism is that it is the only philosophy that does not make the success of science a miracle (Putnam (1975))". Realists roughly argue that, if scientific theories bear no similarity to some parts of reality, then the successes of science would be inexplicable. The anti-realists, like Van Fraassen and Laudan, have defended themselves along various lines. Laudan, for instance, denies that the actual history of the sciences shows a course of progress. He scrutinizes Putnam's argument, and constructs "a confutation of convergent realism."[4] Laudan challenges realists to come up with a definition of convergent realism according to which the better (false) theory guarantees more lasting success. Following Kuipers, I shall refer to this adequacy condition with the term *Laudan's challenge*.[5] Despite the force of Putnam's argument, and despite the realistic origin of my formal work in the following chapters, I do not straightforwardly favour realism, and the present work is not relevant for realists only. Anti-realists also use partial orderings of theories, although they rather call a better theory more *empirically adequate* than closer to the truth.

History teaches that quarrelling about philosophical intuitions is an unsuccessful philosophical method. Comparison of (formal) results often provides a more important contribution to a philosophical debate. The approach-to-the-truth project is no exception. After the collapse of Popper's definition the discussions about verisimilitude intuitions have brought us a plethora of approach-to-the-truth definitions. In this multitude of proposals the content likeness distinction turns out to be the most important one. The comparison of the formal proposals in Chapters 2–3, eventually shows that there are at least two different intuitions hiding behind Popper's informal discussion of verisimilitude; a fact difficult to discover without the formal explanations.

In sum, the approach-to-the-truth project falls within the attempts of realists to define scientific progress, and some authors use verisimilitude to explain lasting empirical success. This does not mean, however, that the results of their endeavours are of no concern for anti-realists.

1.1.2. Popper

Popper's ideas about verisimilitude can be interpreted to be a possible version of convergent realism. Popper tries to formulate an answer to a combination of the following questions which have vexed philosophers for a long time.

How can it be that, on the basis of singular observation, scientists are able to formulate abstract hypotheses that successfully explain and predict phenomena? How do the non-observable terms function in those hypotheses?

We shall come to Popper's proposal in the next section. Here, we only need an outline of his answer. Inspired by Tarski's truth definition, Popper proposes that successful theories are in a way *similar to Tarski's truth* relative to some language. When some relatively successful theory is replaced by a successor, there are indications that this successor is *more similar* to the truth and, hence, to some aspect of reality; thus, it will remain more successful. Popper's answer refers to at least three issues that need further explanation.

The first issue concerns the *concept of truth*. Popper is convinced that Tarski's definition of truth can be extended to the natural language of scientists. Such an extension would avoid the metaphysical bias that accompanied older truth definitions which ascribed truth to ideas instead of sentences. Popper considers Tarski's truth concept perfectly fit to serve as an explicans in his theory of verisimilitude. Consequently, his verisimilitude definition depends on some conceptual framework $\mathscr{L}$ used to specify the theory and the problems to which it is an answer.[6] In the verisimilitude discussion, the truth corresponds to the strongest $\mathscr{L}$-proposition that is true.[7] For those who maintain that this notion of truth implies that there must be in some way "theories in nature", perhaps the following description is more acceptable. First, a language $\mathscr{L}$ is *semantically determinate* (Niiniluoto's term) if all empirical sentences are to receive a definite truth-value.[8] Then, the true theory correctly divides these $\mathscr{L}$-sentences into a set of (Tarski) true sentences and a set of (Tarski) false ones.[9] Verisimilitude is a relation between linguistic (conceptual) elements, and Popper's strategy is an indirect, and sophisticated form of convergent realism. It does not directly claim that successive theories are more like some aspect of *reality* than other theories, but they are more like reality *as described by some language* $\mathscr{L}$, and the relevant background knowledge.

The explanation of the property "being more like the truth" is the second part of Popper's answer that needs elaboration. In other words, the basic idea of one theory being more like the truth than another one, must be defined explicitly. This is the problem of the *definition of verisimilitude*. Popper has given an intuitive description and a formal definition of the concept. Unfortunately, the latter failed to capture Popper's intuitions. This will be the subject of the next subsection. In Chapters 2–3 I deal with several alternative proposals to define verisimilitude which can be reconstructed as adaptations of Popper's original definition. The problem of the definition must emphatically be distinguished from the *epistemic problem* of verisimilitude; and this distinction is the third element of Popper's answer that needs elaboration.

The definition of verisimilitude gives an answer to the question about what *we mean* if we claim that a theory is more verisimilar than another one. The epistemic problem of verisimilitude, instead, reads, *how do we know* that a theory is more verisimilar than another one; or on what indications are we to conclude that a theory is closer to the truth than its predecessor.[10] The fourth chapter contains an evaluation of two important solutions to this epistemic problem of verisimilitude found in the literature. Obviously, Popper's own proposal must be connected with his ideas about falsification and corroboration, although, curiously enough, Popper introduced his methodological recommendations decades before he formulated his ideas about verisimilitude.

1.1.3. Related Concepts

This subsection concerns the difference between verisimilitude and many-valued logic, or even approximate truth. Although the latter concept is related to the idea of verisimilitude, Popper claims that his proposal differs from, and even in some respects improves many-valued logic.[11]

Kleene's introduction to three-valued logic is based on the following idea.[12] It often happens that we have to reason in a situation in which not all sentences have a fixed truth-value for us. A system of three-valued logic prescribes how we are to reason in such a situation. For instance, if φ is true and ψ is unknown, we can safely infer that $\varphi \vee \psi$ is true, and that the truth-value of $\neg\psi$ is unknown. A third truth-value *I* for "unknown", must be added to the system, and the truth-tables of the logical connectives have to be extended. These extensions depend on the interpretation of *I*. J. Łukasiewicz, and D. A. Bochvar give other interpretations to *I* than Kleene, viz. "possible but not necessary", and "meaningless", respectively. Although their truth-tables for the negation are identical, the truth-tables of the other connectives diverge. For instance, according to Łukasiewicz and Kleene, if ψ is false and φ is *I*, then their conjunction is also false, whereas according to Bochvar the conjunction is meaningless. When a three-valued system is defined, there is no need to stick to only one extra truth-value. We can introduce a whole sequence of linearly ordered truth-values, and the higher the truth-value, the more a sentence is apt to be true.

Popper claimed that his ideas about verisimilitude differ from many-valued logic. Even in a two-valued system, two sentences with the same truth-value can be more or less similar to the truth. More specifically, if we know by falsification that two theories are false, one of them can definitely be more like the truth than the other. For instance, the mechanics of Descartes and those of Newton are both false, but we have strong indications that the second is more similar to the truth than the former. Verisimilitude has nothing to do with many-valued logic as "various degrees of truth".

The distinction between approximate truth and verisimilitude is less clear. Both concepts are used in attempts to vindicate some form of convergent realism.[13] The leading idea behind the approximate truth concept is that many useful theories are strictly speaking false, but, within a reasonable margin of error, they are quite acceptable or approximately true. For instance, the Boyle-Gay Lussac law is strictly speaking false, but within some error margin the law gives acceptable results. The same can be said of Newtonian Mechanics. After all, these laws and theories are taught in school. It is common, also among scientists, to distinguish two kinds of falsehoods. Although, strictly speaking, both theories are false, Cartesian mechanics are not on a par with Newtonian Mechanics. Regarding many elementary mechanical questions, and within reasonable margins of error, Newton's laws produce correct answers whereas Descartes mechanics fail completely. Some philosophers of science propose calling Descartes mechanics false and the laws of Newton, approximately true, although they are both strictly speaking false. Often, approximate truth depends on a quantitative measure on the set of possible answers, which also provides a means to decide whether an answer lies within the margins of error. Those who give a fundamentally quantitative approach-to-the-truth definition, as for instance Niiniluoto, will also incorporate the concept of approximate truth. The difference between a quantitative and qualitative approach will be explained in subsection 1.5.2 (p.29).

I end my sketch of the approach-to-the-truth project with an observation for those philosophers of science who think that all phenomena in the scientific process can be explained using probability. The verisimilitude approach is useful in the case in which we already know that a theory is false. In the Bayesian approach, and in Carnap and Hintikka systems, falsified theories receive zero probability. This deprives us of the means by which we use between two falsified theories.

1.2. POPPER'S PROPOSAL AND ITS FAILURE

As we saw, the idea underlying the verisimilitude proposals is to vindicate a form of convergent realism; it is to be distinguished from many-valued logic and approximate truth; but what is the positive idea of the verisimilitude notion? In the famous tenth chapter of *Conjectures and Refutations* (1963), the last one of the *Conjectural* part, Popper reveals his intuitions about, and defines the notion of verisimilitude.

1.2.1. Verisimilitude Intuitions

Popper (1963) suggests in the tenth section of the tenth chapter

...that we combine here the ideas of truth and of content into one—the idea of a degree of better (or worse) correspondence to truth or of greater (or less) likeness or similarity of truth; or to use a term already mentioned above (in contradistinction to probability) the idea of (degrees of) *verisimilitude* (Popper (1963) pp. 232-233, his italics).

In the same section Popper explains his views on the important role of the logical content when theory t_2 supersedes t_1 "in the sense that t_2 seems—as far as we know—to correspond better to the facts than t_1." He gives "a somewhat unsystematic list of six types of" such cases.

"(1) t_2 makes more precise assertions than t_1, and these more precise assertions stand up to more precise tests.
(2) t_2 takes account of, and explains, more facts than t_1 (which will include for example the above case that, other things being equal, t_2's assertions are more precise).
(3) t_2 describes, or explains, the facts in more detail than t_1.
(4) t_2 has passed tests which t_1 has failed to pass.
(5) t_2 has suggested new experimental tests, not considered before t_2 was designed (and not suggested by t_1, and perhaps not even applicable to t_1); and t_2 has passed these tests.(6) t_2 has unified or connected various hitherto unrelated problems. (*ibid.* p. 232)."

The following subsection contains an explanation of how Popper's formal verisimilitude definition combines his ideas about truth and content.

1.2.2. The Definition

Popper (1963) presents his verisimilitude definition in the eleventh section of chapter ten where he uses a basic notion of improvement to combine content and truth into the idea of "better correspondence to the truth". Generally, common sense prefers *B* to *A*, if *B*'s advantages exceed those of *A*, and *A*'s drawbacks are worse than those of *B*. Popper's verisimilitude definition—just as all its successors—is based on this philosophical intuition:

(1) The theory ψ is better than φ if the merits of ψ improve those of φ, and the drawbacks of φ exceed those of ψ.[14]

Popper takes the merits of a theory to be all its factually true consequences and the drawbacks all its factually false consequences, where factually true/false is defined using Tarski's truth definition. By doing so Popper claimed he developed means by which all theories can be classified objectively by their share of true and false consequences. Of course, he was aware of the asymmetry between the set of true consequences of a sentence, and the set of false ones.[15] The set of true consequences is deductively closed; i.e. a consequence of any element of the set is again an element of that set. Popper contends that the set of false consequences, "is not, strictly speaking, a 'content', because it does not contain any of the true conclusions of the false statements which form its elements."[16] Notwithstanding this asymmetry, Popper used consequence classes to distinguish between the false and true part of an empirical theory.

To implement the improvement intuition (1), Popper lets a consequence class of an axiom φ represent a theory, and divides it into two parts; the false and the true one. He calls the former the *truth-content* and the latter the *falsity-content* of φ and denotes them by $Ct_T(\varphi)$ and $Ct_F(\varphi)$, respectively:

(2) $Ct_T(\varphi) := \{\psi \mid \varphi \vDash \psi, \text{ and } \psi \text{ is factually true}\}$
(3) $Ct_F(\varphi) := \{\psi \mid \varphi \vDash \psi, \text{ and } \psi \text{ is factually false}\}$[17]

In the sequel $\mathrm{Cn}^T(\varphi)$ and $\mathrm{Cn}^F(\varphi)$ represent the set of true and false φ-consequences, and in the remainder of this chapter, except where I explicitly claim the opposite, the truth is assumed to be complete.

Popper's formal definition is somewhat weaker than the intuition "more true statements and fewer false ones." Improvement of a theory consists in increase of its truth-content or decrease of its falsity-content. Assuming the completeness of the truth, I paraphrase the reflexive companion of Popper's definition as follows,

DEFINITION 1.1: According to Popper, the *theory* ψ *is at least as verisimilar as* φ iff
a. $Ct_T(\varphi) \subseteq Ct_T(\psi)$
b. $Ct_F(\psi) \subseteq Ct_F(\varphi)$
Notation: $\psi \leq^P \varphi$

Regarding true theories, Popper's definition clearly favours strong theories to weak ones; moreover a false theory never improves a true one, as the falsity-content of the latter is the empty set. Thus, Popper orders all theories by their truth-value and logical strength.

To stress the asymmetry in Popper's definition, and to improve our understanding of the falsity-content, I introduce the dual of Tarski's famous consequence class, the class of *antecedences*.[18] In the next definition $\mathrm{Prop}(\mathcal{L}) := \{[\varphi]_\equiv \mid \varphi \in \mathrm{Sent}(\mathcal{L}))\}$ designates the set of all $\mathcal{L}$-propositions.

DEFINITION 1.2: Let $\mathcal{L}$ be a formal language. The set of *antecedences* of φ is defined by $\mathrm{An}(\varphi) := \{\chi \in \mathrm{Prop}(\mathcal{L}) \mid \chi \vDash \varphi\}$

The properties of the class of antecedences mirror those of the class of consequences. In fact, $\mathrm{An}(\varphi)$ may be defined by a consequence class: $\chi \in \mathrm{An}(\varphi)$ iff $\neg\chi \in \mathrm{Cn}(\neg\varphi)$. Furthermore, due to the definition of "$\vDash$" $\mathrm{An}(\varphi)$ is closed with respect to the antecedence relation, as $\mathrm{Cn}(\varphi)$ is closed with respect to the consequence relation.

(4) $\forall\chi \in \mathrm{Cn}(\varphi)$, if φ is *true* (*false*), then χ is *true* (*true or false*)
(5) $\forall\chi \in \mathrm{An}(\varphi)$, if φ is *false* (*true*), then χ is *false* (*true or false*)

If the truth is finitely axiomatizable by τ, then we let $Cn(\tau)$ designate the set of true propositions of $\mathscr{L}$, and let $An(\neg\tau)$ represent the set of false propositions. As a result, the assumption of a complete truth may be paraphrased as follows:

(6) $\forall\chi \in Prop(\mathscr{L})$: $\chi \in Cn(\tau)$ or $\chi \in An(\neg\tau)$

Popper did not take into account that the set of all false propositions is closed with respect to the antecedence relation (see (5)). An analysis using antecedence classes and consequence classes reveals the peculiar asymmetry of Popper's proposal. If we substitute (2) and (3) in the original definition, and erase redundancies, then we obtain:

DEFINITION 1.1′: According to Popper ψ is at least as close to the truth τ as φ iff
a′. $Cn(\varphi) \cap Cn(\tau) \subseteq Cn(\psi)$
b′. $Cn(\psi) \cap An(\neg\tau) \subseteq Cn(\varphi)$
Notation: $\psi \leq^P_\tau \varphi$

In Chapter 2, we shall see that an important revision of Popper's verisimilitude proposal, the symmetric difference definition of Miller and Kuipers, substitutes $Cn(\neg\tau)$ for $An(\neg\tau)$ in the second or *falsity* clause.

1.2.3. The Failure

We now come to the well-known collapse of Popper's definition. The next proposition shows that for two factually false theories φ and ψ, definition 1.1 applies only if φ and ψ are equivalent (Figure 1).

PROPOSITION 1.1: Suppose φ and ψ are false; then $\psi \leq^P \varphi$ iff $\psi \equiv \varphi$

Proof: $\Leftarrow$: Clearly, if $\psi \equiv \varphi$, then $\psi \leq^P \varphi$. $\Rightarrow$: Suppose that $\psi \leq^P \varphi$. By definition this assumption implies 1. $Cn^F(\psi) \subseteq Cn^F(\varphi)$, and as $\psi \in Cn^F(\varphi)$ this means that $\varphi \vDash \psi$ (†). The other part of the definiens reads 2. $Cn^T(\varphi) \subseteq Cn^T(\psi)$. Since ψ is false, $\neg\psi$ is true; and as $\varphi \vee \neg\psi$ is a true consequence of φ, 2. implies $\psi \vDash \varphi \vee \neg\psi$. Therefore, by propositional logic $\psi \vDash \varphi$ (‡); and the conjunction of (†) and (‡) equals $\psi \equiv \varphi$. ⊠

Consequently, Popper's own irreflexive companion, $<^P$, relates no false sentences. Obviously, the proposition is fatal for Popper's proposal, since all empirical theories are strictly speaking false.[19] It was C. Hempel who was the first to give a reported proof of the uselessness of Popper's proposal. According to Hattiangadi, Hempel produced a refutation of Popper's verisimilitude definition at the defence of his doctoral dissertation in June 1970.[20] In 1974, Tichý and Miller both published a proof of a theorem of the same purport, which they found independently.[21]

EXAMPLE: I use a two propositional language $\mathscr{L}[p,q]$ to illustrate the failure of Popper's definition. Despite its three-dimensional appearance, Figure 1 is the (Hasse-) diagram of the $<^P_{p\wedge q}$-ordering of Prop($\mathscr{L}[p,q]$). If $\psi <^P_{p\wedge q} \varphi$, then ψ is depicted below φ, and a line connects them. Due to transitivity, we do not need to draw lines between pairs $\langle\psi, \chi\rangle$ that have an intermediate.[22]

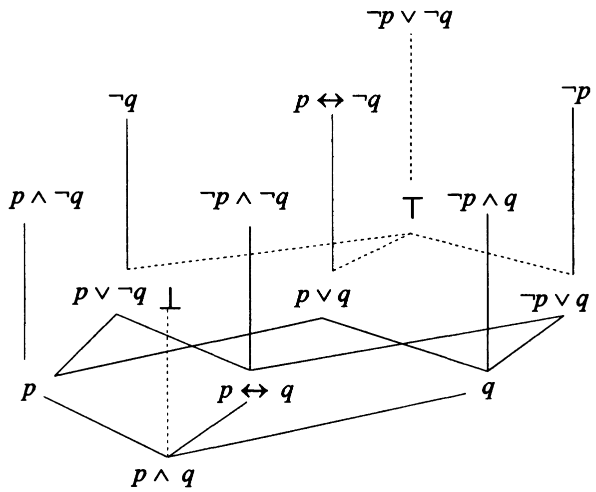


Fig. 1. $<^P_{p\wedge q}$-ordering of Prop($\mathscr{L}[p,q]$).

Note that no single line connects two false propositions; Popper's definition fails to order them. Furthermore, no proposition in EmProp($\mathscr{L}$) (:= Prop($\mathscr{L}$)-$\{\top,\bot\}$) is comparable to $\neg p \vee \neg q$ (consider the continuous line segments in the Hasse-diagram). Only if we add the contradiction $\top$ and $\bot$ to our considerations, connected by the dotted lines, $\top$ is better than $\neg p \vee \neg q$, and the truth is better than $\bot$. Finally, the diagram illustrates that Popper's definition cannot be used to compare propositions of the same logical strength and truth value; for instance, the verisimilitude of $p \vee q$ and $p \vee \neg q$ are uncomparable (even $p \nleq^P_{p\wedge q} \neg p$!). *End Example*

1.2.4. Analysis of the Failure

The cardinal mistake of Popper's original verisimilitude proposal is the idea that the truth-content of a false sentence φ is independent of its falsity-content. The following proposition proves the contrary: the *truth-contents* of false sentences *vary with their falsity-contents*.

PROPOSITION 1.2: Let φ and ψ be false; then (1) $Cn^T(\varphi) \subseteq Cn^T(\psi) \Leftrightarrow$ (2) $Cn^F(\varphi) \subseteq Cn^F(\psi)$

Proof: $\Leftarrow$: The falsity of φ and ψ, and (2) together imply $\psi \models \varphi$; therefore (1) obtains. $\Rightarrow$: We shall give an indirect prove. Assume (1) holds; then suppose there is a $\chi \in Cn^F(\varphi)$ that is not a consequence of ψ. As $\neg\psi$ is true, and $Cn(\varphi)$ deductively closed, $\chi \vee \neg\psi \in Cn^T(\varphi)$. Since $\chi \notin Cn(\psi)$, $\chi \vee \neg\psi \notin Cn^T(\psi)$. This contradicts assumption (1); and hence (2). ⊠

Popper's assumptions about the truth-content and falsity-content imply that increase of the true part of a theory is the same as increase of its false part. Consequently, it is impossible to improve a false theory according to Popper's original proposal. It is easy, however, to correct Popper's error. Obviously, we may drop the entire falsity clause altogether. Popper also thought about this way of repairing his proposal;[23] and we shall come to this proposal in Section 1.4, and use it extensively in Chapter 6. A second way to save Popper's proposal is to drop the assumption that the theories are *deductively closed*. The argument of Miller and Tichý fails if the union of truth- and falsity-content of a theory is not deductively closed; and nothing blocked an independent comparison of the truth- and falsity-contents. (see Appendix 1).

The final way to break the interdependence between the truth- and falsity-content of φ is to change their mathematical objects of application. For example, Miller and Kuipers have proposed to paraphrase the idea of "more verities and fewer falsities" in terms of *models* of the truth and models of the negation of the truth, respectively. In Chapter 2 we shall see that this proposal does not suffer from the flaw of Popper's definition. Interestingly, the proposal of Miller and Kuipers is the symmetric counterpart of definition 1.1′. Rephrased in terms of consequence classes it reads: ψ is at least as verisimilar as φ iff

a. $Cn(\varphi) \cap Cn(\tau) \subseteq Cn(\psi)$
b. $Cn(\psi) \cap Cn(\neg\tau) \subseteq Cn(\varphi)$

If Popper had taken the asymmetry in his proposal seriously, he might have anticipated the mistake covered up by it. I shall present Kuipers's structuralist proposal in Chapter 2.

1.2.5. Popper Saved by Non-Monotonic logic

A philosophically more profound analysis of the collapse of Popper's definition than the technical ways of repair of the preceding subsection is the following. Popper's definition breaks down since he uses *monotonic means* to paraphrase fundamentally *non-monotonic intuitions* about theory improvement. According to

Popper we improve a theory if we *add* a new axiom that extends the set of true consequences, and that *diminishes* the set of false consequences.

Non-monotonic reasoning is a branch of philosophical logic that originates in artificial intelligence.[24] A non-monotonic deductive system contains, besides classical logical notions, an alternative, non-monotonic deduction operator. A non-monotonic consequence of a proposition is not a truth functional inference, but a consequence that is assumed to hold by default. New relevant information may cause the withdrawal of that consequence. This inference relation is designated by the $\rightarrow$ symbol. An example to be found everywhere in the literature reads:

(7) "Tweety is a bird" $\rightarrow$ "Tweety can fly"
(8) "Tweety is a bird and Tweety is a penguin" $\rightarrow$ "Tweety cannot fly"

In this example, the extra information that "Tweety is a penguin" is the reason for the withdrawal of the consequence (7).

There is a direct link with Popper's definition.[25] Suppose χ is an factually true law, and φ is a false theory that does not imply χ. Then, $\varphi \wedge \chi$ has more true consequences than φ; and regarding their truth-contents, $\varphi \wedge \chi$ improves φ. Regarding the false consequences, however, Popper wants $\varphi \wedge \chi$ *to withdraw* some false consequences of the falsity-content of φ. Schematically:

(9) $Ct_T(\varphi \wedge \chi) \supset Ct_T(\varphi)$ and possibly
(10) $\exists\alpha \in (\mathrm{Cn}(\varphi) \cap \mathrm{An}(\neg\tau)): \alpha \notin \mathrm{Cn}(\varphi \wedge \chi)$

Clearly, this is a non-monotonic requirement, since to *add* an extra axiom χ causes the *withdrawal* of some (false) consequences. Consequently, we may pursue another way to save Popper's definition. I define the *default consequences* and the *default falsity-content* of a proposition φ as follows.

$$\mathrm{Cn}^{\rightarrow}(\varphi) :=_{def} \{\chi \in \mathrm{Sent}(\mathcal{L}) \mid \varphi \rightarrow \chi\}$$
$$Ct_F^{\rightarrow}(\varphi) :=_{def} \mathrm{Cn}^{\rightarrow}(\varphi) \cap \mathrm{An}(\neg\tau),$$

This definition does not exclude the combination of $Ct_F(\psi) \supset Ct_F(\varphi)$, and $Ct_F^{\rightarrow}(\psi) \subset Ct_F^{\rightarrow}(\varphi)$. We may reformulate Popper's original proposal as follows:

DEFINITION 1.3: The theory ψ is *non-monotonically at least as verisimilar as* φ iff $Ct_T(\varphi) \subseteq Ct_T(\psi)$ and $Ct_F^{\rightarrow}(\psi) \subseteq Ct_F^{\rightarrow}(\varphi)$
Notation: $\psi \leq_\tau^{\rightarrow} \varphi$

If ψ and φ are both false, then the definition claims that $\mathrm{Cn}^{\mathrm{T}}(\psi) \supseteq \mathrm{Cn}^{\mathrm{T}}(\varphi)$ (which implies $Ct_F(\varphi) \subseteq Ct_F(\psi)$) and $Ct_F^{\rightarrow}(\psi) \subseteq Ct_F^{\rightarrow}(\varphi)$. This is not as farfetched as it may seem at first sight. New, and more extensive theories make more explicit cognitive claims about the subjects than the old theory. Consequently, there is a risk more mistakes will be made; however, if some default answers of the old theory to relevant questions are false, then the new theory may cancel some of these failures by *adding* a new principle. In this sense the new theory will contain fewer falsehoods than the old one. According to Newtonian Mechanics, for instance, there is by default no limit to the growth of velocity of particles. Special Relativity, however, adds an axiom about maximum speed to the (adjusted) Newtonian laws, which excludes false consequences of Newtonian Mechanics regarding super Luminal velocities. At the same time Special Relativity preserves the appropriate answers of Newtonian Mechanics. To conclude, Popper's failure was to erroneously use monotonic means to paraphrase a non-monotonic desideratum.[26] Historically, no proposal dropped the monotonicity assumption; and in the sequel, I shall not resume this line of repair.

In the present section, we considered Popper's verisimilitude proposal and its failure. Popper's original starting points were:

1. Popper used classical, *monotonic logic* to define his verisimilitude notion
2. theories are *deductively closed*
3. the better theory has the *larger truth- and lesser falsity-content*
4. truth-content of a theory is the set of *all* its true consequences; its falsity-content the set of *all* its false consequences

We have seen that modification of only one of these restrictions may save Popper's original intentions. In Section 1.4, we shall return to the two prevailing strategies of revision found in the literature. First, the next section concerns the question: what is the most appropriate way to compare the diversity of proposals that followed the breakdown of Popper's definition?

1.3. COMPARISON OF THE DEFINITIONS

After the collapse of Popper's definition, the verisimilitude notion enjoyed an increase in interest. Several philosophers proposed new explanations that did not suffer from the drawback of Popper's definition. This did not prove to be too difficult, and today, after more than thirty five years there is a proliferation of approach-to-the-truth definitions.[27] Unfortunately, their form and content diverge considerably, and an neutral survey of the actual proposals does not exist, and the question of how to compare or even assess the different proposals is a hard one. The evaluation of formal explications that deal with prescientific, philosophical intuitions has two sides. In the first place, we must compare the *formal proposals* secondly, but equally important, the comparison concerns the *interpretation* of these formal proposals. In the verisimilitude discussion, there are considerable differences of opinion on both aspects; this will become clear in Chapter 2. First, I shall introduce the way how I have compared the various approach-to-the-truth proposals.

1.3.1. The Definition; its Application and Interpretation

Comparison of approach-to-the-truth definitions in the literature has primarily been focussing on the (dis)agreements of intuitions. Witness the passionate discussions without any comprehensive conclusion, this method has not proved to be fruitful. The verisimilitude notion seems to incorporate too many different philosophical intuitions. Another method of diagnosis is a theoretical one, in which the metatheoretical properties of the definitions are systematically compared.[28] Although it yields some results, it also leads to difficulties as the various proposals use incompatible starting-points. A common ground for the most important proposals is hard to find as they do not share the same level of abstraction. Qualitative considerations alternate with quantitative ideas, and important advantages and drawbacks for one explanation are perhaps minor, or even irrelevant, issues for another. In brief, the verisimilitude debate is too heterogeneous to admit a theoretically sophisticated evaluation. I have found a middle course between intuitions and metatheory and distinguish between the following three issues,

1. the *formal definition*
2. the *mathematical application*
3. the intuitive *interpretation*

As for the first subject, all proposals include a formal approach-to-the-truth definition. I shall give an elaborate exposition of several prominent proposals in Chapters 2–3.

The "mathematical application" of the definition is the set of elements that are ordered by the definition. All researchers are rather explicit about their mathematical applications. For instance, they all apply their definitions to finite propositional languages, or to first order languages. Several proposals also incorporate quantitative applications such as distances for vector fields (subsection 1.3.2). In my survey of the different proposals I compare the similarities and differences of the formal orderings of one particular elementary mathematical application, viz. *the (finite) propositional language.* The result is that the contrast between the *content* and *likeness* definitions, which, among many others, already existed in the literature, proves to be the most fundamental one. Subsection 1.4.2 contains a definition of the general features of this distinction.[29]

Although the different researchers are rather explicit about their definition(s) and application(s), the interpretation of the proposals is often left to the reader. I mean by "the interpretation" the way in which the proposal connects scientific laws, evidence, and theories to the formal elements of the definition. Thus, the interpretation prescribes how the elements of the formal system represent the principal parts of real-life science. This issue gains momentum, as differences in the interpretation explain why researchers disagree about the best way to revise

Popper's proposal. We shall encounter the link between this interpretation issue and the difference between likeness and content definitions in Section 1.4. First, I shall present the variety of mathematical applications used by the definitions we will encounter in Chapters 2–3.

1.3.2. Mathematical Applications; the Lindenbaum Algebra

The mathematical application of an approach-to-the-truth definition is the set of elements that are ordered by the definition. This application is clearly distinct from the definition itself. Thus, Niiniluoto investigates many truthlikeness measures without being specific about the mathematical applications. Independently, he enumerates a wealth of possible mathematical applications of his approach.[30] Not all proposals keep the formal definition and the application apart that rigorously. Popper applied his proposal to (sets) of statements and numerical statements. Hilpinen's explanation is about models. Brink and Heidema apply their definitions to propositional languages. Thus, the situation must be described as follows. Although, in principle, an approach-to-the-truth definition is independent of its mathematical objects of application, often, both are intimately connected. A thorough comparison of the various proposals must take this feature into account. The following text displays the variety of applications to be found in the literature, and provides applications for every definition presented in the next chapter.

As we saw, *Popper's proposals concern primarily formal statements,* he wanted to prove, using his verisimilitude explanation, that scientists should strive towards theories with a low probability. That was the reason behind Popper extending his proposal to quantitative terms, which suffered form the same pitfall as his qualitative definition.[31] Popper thought that point estimation must also be related to verisimilitude. For instance, he considered the statement "it is 9.45 p.m." closer to the truth than "it is 9.40 p.m." if the true statement reads "it is 9.48 p.m.."[32] Those who think this is obvious must read Miller (1975) where the author argues that all incorrect point-estimates are equally wrong.[33] *Miller's* definition orders

a. elements of normal Boolean Algebra's and
b. elements of normal Brouwerian Algebra's.[34]

Miller's scepticism about the order of quantitative statements causes him to avoid this application. According to *Oddie*, a mature approach-to-the-truth definition must incorporate:

a. propositions
b. distances between elements of $\mathbb{R}$, and distances between intervals of $\mathbb{R}$
c. kinds of individuals and structures for polyadic languages[35]
d. ideally, weighted predicates.

After the breakdown of Popper's proposal, *Hilpinen* was the first to call his proposal a truthlikeness definition. He preferred the term truthlikeness to verisimilitude for reasons that will become clear in the next chapter. His explanation is based on similarity between possible worlds.[36] *Niiniluoto* took over the basic idea of Hilpinen but did not want to base his explanation on an unanalysed similarity notion. Similar to the proposals of Tichý and Oddie, Niiniluoto prefers quantitative measures of truthlikeness to qualitative ones. As mentioned earlier, his sophisticated approach distinguishes between the definition of the measures, and the instantiations of those abstract measures. His treatise *Truthlikeness* contains a wealth of examples. All mathematical structures that fit Niiniluoto's cognitive problem may serve as an application. The examples comprise:

a. elements of $\mathbb{R}$
b. intervals of $\mathbb{R}$
c. functions from $\mathbb{R}^{n-1}$ to $\mathbb{R}$
d. Carnapian Q-predicates
e. constituents of monadic languages
f. nomic (modal) constituents
g. depth-d constituents of polyadic first-order languages
h. complete first-order theories[37]

Kuipers presents his approach-to-the-truth project in structuralist terms. Within his framework, Kuipers uses modal terminology and intuitively distinguishes between the physically possible and impossible structures. He mentions several applications of his abstract definition.[38] Among them are:

a. Sneed/Suppes structures with some similarity type
b. sentences of a formal languages; first order structures
c. real number structures

The refined version of Kuipers's definition also uses idealization and concretization triples.[39] Finally, more recently *Heidema*, in cooperation with *Brink* and *Burger*, has proposed a truthlikeness explanation based on power relations, they base their definition entirely on propositional languages.[40]

A diverse set of mathematical applications was presented in the preceding paragraph . Examining the lists, we see that the only application all proposals deal with is the *finite propositional language*. This enables us to use this well-understood formal system as an instrument of calibration for all approach-to-the-truth proposals. This proves to be a strong heuristic; since if there is already disagreement between definitions about this elementary case, then the prospect of a useful comparison of more sophisticated situations looks slim.

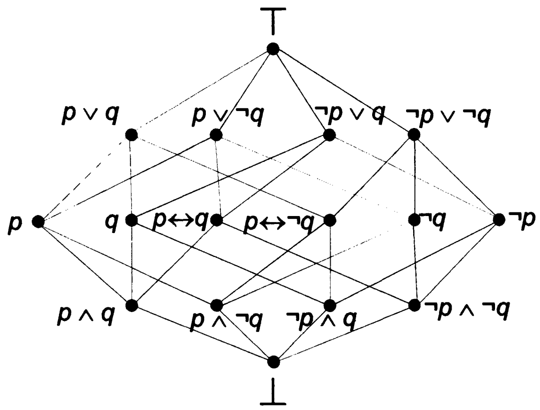


Fig. 2. Lindenbaum Algebra of $\mathscr{L}[p,q]$

In the present monograph, I shall often represent a propositional language by its corresponding *Lindenbaum Algebra.* A Lindenbaum algebra is a Boolean algebra whose elements are propositions of a finite propositional calculus—i.e. sets of logically equivalent sentences.[41] The lowest upper bounds and the greatest lower bounds in the algebra are the disjunction and conjunction of the language, respectively; and the complement represents the negation. This algebra plays an important role in various proposals found in the literature. In this publication I shall compare the various approach-to-the-truth proposals by the way they order the elements of a finite Lindenbaum algebra.

1.3.3. Interpretations: Laws and Data in the Lindenbaum Algebra

The importance of the *intuitive interpretation* of the mathematical applications is stressed in this subsection. As stated above, the interpretation is the way in which the application represents real *theories* and *empirical data.* Up to this moment, this facet of interpretation has escaped the attention of most investigators. We shall see that some differences of opinion about the definitions are due to the different assumptions made about the interpretations.

As we saw, the applications that will be studied the most thoroughly in the chapters that are to come are finite propositional languages. More specifically, then, our interpretation question reads: how are the propositional languages related to real life theories and empirical data? The next example illustrates the disagreement between the various researchers regarding interpretation even in this most elementary case.

EXAMPLE: Let the mathematical application of a definition be a propositional language $\mathcal{L}[p,q,r]$. Of course, there are infinitely many syntactically different sentences in this language, but we shall concentrate on the propositions of $\mathcal{L}$, Prop($\mathcal{L}$), the sentences of $\mathcal{L}[p, q, r]$ modulo logical equivalence. Generally, if the number of atomic propositions is n, then $|\text{Prop}(\mathcal{L})| = 2^{2^n}$. A proposition $\alpha \in \text{Prop}(\mathcal{L}[p_1, \ldots p_n])$ is complete and is called a *constituent* if it has the following form

$$(11) \qquad \bigwedge_{i=1}^{n} (\pm)p_i$$

in which $(\pm)p_i$ is a *literal*—i.e. a negated or affirmed atomic proposition. There is no noncontradictory proposition of $\mathcal{L}$ different from α that implies α. Since α is *complete*, for all $\varphi \in \text{Prop}(\mathcal{L})$ $\varphi \in \text{Cn}(\alpha)$ or $\varphi \in \text{An}(\neg\alpha)$ obtains. Cnst(φ) designates the set of all constituents that imply φ. Suppose the complete truth τ of $\mathcal{L}$ is the constituent $p \wedge q \wedge r$; then we may ask whether (12) or (13) is closer to the truth τ.

$$(12) \qquad \neg p \vee \neg q \vee \neg r \quad (\equiv \neg(p \wedge q \wedge r))$$
$$(13) \qquad \neg p \wedge \neg q \wedge \neg r,$$

In due course, we will see that there are two groups of approach-to-the-truth proposals. In the orderings of one group (12) is more similar to $p \wedge q \wedge r$ than (13); according to the second group it is the other way around. *End Example*

Many participants in the discussion agree with Popper's original intuition that a better theory has more merits and fewer drawbacks than the worse one. Why, then, can such an elementary example as the preceding one cause such difference of opinion? This difference is due to the different ways in which the various researcher paraphrase real theories and evidence in $\mathcal{L}$. Suppose that theories and evidence can be reformulated as propositions of a finite propositional language $\mathcal{L}$, the number of atomic sentences of which may be fairly large. Then there are at least two different ways to interpret this.

According to the first interpretation literals are taken to be *highly abstract* scientific hypotheses. Consequently, they are only indirectly falsifiable or verifiable, and therefore not the literals, but their consequences are basic (or observable) statements. The three laws of Newton serve as an example. Evidence is accounted for by the set of all logical consequences of the theory. The basic statements correspond with relatively large *disjunctions* of atomic propositions or even to sentences of an observable sublanguage of $\mathcal{L}$, in the case of stratified theory representation.[42] Moreover, the basic statements consists of observable laws instead of descriptions of individual experiments. To stay with Newtonian mechanics, Kepler's laws serve as an example of relevant observational consequences. According to this interpretation of $\mathcal{L}[p,q,r]$, $\varphi := p \wedge \neg q \wedge r$ is an example of a theory, and $\delta := \mathrm{Prop}(\mathscr{L}_o) \cap \mathrm{Cn}(\varphi)$ an example of empirical data. This interpretation is in line with Popper's approach.

The second possible interpretation paraphrases the theory φ as a (relatively) large *conjunction* of literals all of which are true according to φ. These literals are mainly basic (or observable) statements, and not sentences of a fairly high level of abstraction. Evidence is accounted for by the consequences of the theory that are literals. If φ explains all data gathered thus far, then the set of literals that represents the empirical data is a subset of φ. To be more explicit: in $\mathscr{L}[p_1, \dots p_{100}]$ a serious theory φ asserts about most p_i whether it is true or false. Examples of a falsified law φ and data δ are, respectively:

$$(14) \qquad \varphi := \bigwedge_{i=1}^{80} p_i \qquad \delta := p_1, \neg p_{11}, p_{13}, \neg p_{19}$$

Examples become more interesting, of course, if the number of literals is large. Obviously, the *formal* differences of the two approaches concerns the syntactic form of the data δ rather than that of the theory φ.

A good illustration of this way of interpretation can be found in Tichý (1974) and Miller (1974). The authors discuss a three propositional language $\mathscr{L}[h,r,w]$ the atomic propositions of which designate 'hot', 'rainy', and 'windy', respectively. We shall consider the issue extensively in the Chapter 5.[43]

Summarizing the present subsection, we see a differences of opinion about the status of propositional languages. The first possible interpretation paraphrase theories as (conjunctions of) literals of a rather abstract nature, and *all* formal consequences are supposed to correspond with relevant empirical data and are equally important. The second possible interpretation paraphrases theories as large conjunctions of (relative) observable statements, and empirical data are conceived as sets of literals, which form a relatively small subset of all logical consequences of theories. The two possible interpretations foreshadow the content likeness distinction defined in the next section.

Let me summarize the present section. When comparing the definitions, we shall concentrate on the most elementary linguistic application, viz. the finite Lindenbaum algebra. Furthermore, we shall keep an eye on the way scientific laws, theories, and empirical data are interpreted in terms of that algebra. Differences in interpretation become relevant in Chapter 4, which concerns the epistemic problem of verisimilitude.

1.4. DISTINCTION BETWEEN CONTENT AND LIKENESS PROPOSALS

This section concerns the distinction between *content* and *likeness* definitions, the paramount distinction between two groups of definitions found in the literature. In the first subsection the likeness-content contrast is introduced as a difference of opinion about the revision strategies of Popper's definition. Then, in the second subsection the *intuitive* distinction between likeness and content proposals are intuitively explained; in the third subsection a *formal* definition of the distinction is given. Finally, in the last section an argument against likeness, and an argument against content proposals are presented.

1.4.1. Likeness and Content Strategies of Revision

In this subsection a rational reconstruction of the two important ways to revise Popper's proposal is given; this foreshadows the differences between likeness and content definitions. To keep things as simple as possible, we shall concentrate on a finite propositional language $\mathcal{L}$ equipped with a complete truth τ.

The first revision strategy of Popper's original proposal holds on to the first intuitive *interpretation* of the Lindenbaum algebra, and eliminates the falsity clause of the definition. I have already hinted at the fact that Popper himself has considered this way of repair. I shall call it the *consequence definition*, or the +-definition; I denote its ordering by $\leq^+$. It plays an important role in Chapter 6.

DEFINITION 1.4: Let τ be the complete truth of $\mathcal{L}$. For ψ and $\varphi \in \mathrm{Prop}(\mathcal{L})$, ψ is *as least as verisimilar as φ* iff $\mathrm{Cn}^{\mathrm{T}}(\psi) \supseteq \mathrm{Cn}^{\mathrm{T}}(\varphi)$.
Notation: $\psi \leq^+ \varphi$

Observation 1.3: For all ψ, φ and complete $\tau \in \mathrm{Prop}(\mathcal{L})$: $\psi \leq^+ \varphi$ iff $\psi \vDash \varphi \vee \tau$.
Proof: $\psi \leq^+ \varphi$ means $\mathrm{Cn}(\psi) \supseteq \mathrm{Cn}(\varphi) \cap \mathrm{Cn}(\tau)$; this equals $\mathrm{Cn}(\psi) \supseteq \mathrm{Cn}(\varphi \vee \tau)$ which means $\psi \vDash \varphi \vee \tau$ ⊠

An important consequence of this proposal is that *all* contingent consequences, literals and complex formulae, are considered *equally* important; this holds for all content proposals. The content revise strategy has, among others, an epistemic underpinning. If only literals are considered relevant, as will be the case in likeness definitions, then we must assess consequences of a theory by their logical form, sorting out the literals and the complex formulae. It is highly questionable whether this is relevant for assessment of empirical data in scientific practice.

Observation 1.4: $\leq^+$ is a preordering on $\mathrm{Prop}(\mathcal{L})$.
Proof: Clearly, $\leq^+$ is reflexive. Since $\mathrm{Cn}(\psi) \cap \mathrm{Cn}(\tau) \supseteq \mathrm{Cn}(\chi) \cap \mathrm{Cn}(\tau)$ together with $\mathrm{Cn}(\chi) \cap \mathrm{Cn}(\tau) \supseteq \mathrm{Cn}(\varphi) \cap \mathrm{Cn}(\tau)$ implies $\mathrm{Cn}(\psi) \cap \mathrm{Cn}(\tau) \supseteq \mathrm{Cn}(\varphi) \cap \mathrm{Cn}(\tau)$, it is also transitive. ⊠

The $\leq^+$-ordering is not antisymmetric. Take, for instance, $\tau := p \wedge q$, $\psi := p$ and $\varphi := p \wedge \neg q$; then $\psi \vDash \varphi \vee \tau$ and $\varphi \vDash \psi \vee \tau$ and $\psi \nequiv \varphi$. Let $\sim^+$ designate the $\leq^+$-equivalence on $\mathrm{Prop}(\mathcal{L})$ and let $\psi \sim_\tau \varphi$ abbreviate $\psi \leftrightarrow \neg\varphi \vDash \tau$.

Observation 1.5: For all $\psi, \varphi \in \mathrm{Prop}(\mathcal{L})$: $\psi \sim^+ \varphi \Leftrightarrow \psi \sim_\tau \varphi$.

Proof: $\psi \sim^+ \varphi$ equals $\psi \leq^+ \varphi$ and $\varphi \leq^+ \psi$; and since $\psi \vDash \varphi \vee \tau$ and $\varphi \vDash \psi \vee \tau$ equals $\neg\varphi \wedge \psi \vDash \tau$ and $\varphi \wedge \neg\psi \vDash \tau$, $\psi \sim^+ \varphi$ equals $\psi \leftrightarrow \neg\varphi \vDash \tau$. ⊠

Figure 3 shows the Hasse-diagram of the $\leq^+$-ordering of Prop($\mathscr{L}[p,q]$)/~. The diagram clearly shows that the removal of the falsity clause brings about an ordering that sometimes favours a false theory to a true one. For instance, $p \wedge \neg q \leq_{p\wedge q} p \vee q$. The $\leq^+$-definition is not "truth-value dependent"; a definition of this notion can be found in the last section.

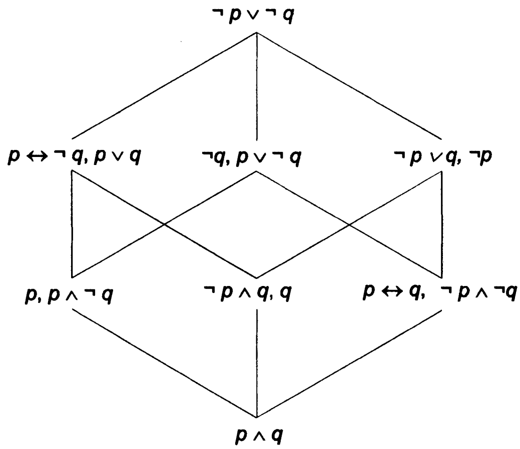


Fig. 3. The $\leq^+$-ordering of Prop($\mathscr{L}[p,q]$)/~.

Another important implication of this revision is that the underlying intuition has changed since the truth-content has overruled the falsity-content as the falsity clause is totally left out of consideration. Instead of "more verities and fewer falsities" the underlying intuition changed into "more verities."[44] In other words, this revision changes the third assumption at the end of subsection 1.2.5, p. 13. The consequence definition foreshadows a general strategy of repair that resulted in *content proposals*. According to this way of revision, $\neg p \vee \neg q$, the negation of the truth $p \wedge q$, is the worst empirical theory possible; it does not have true empirical consequences. The proposition $\neg p \wedge \neg q$, is better than $\neg p \vee \neg q$ since it has many true empirical consequences; and as (relatively strong) true consequences are what scientists are after, they do not appreciate the latter theory. Needless to say that according to this revision strategy the literals represent highly abstract empirical laws and large disjunction of these literals represent empirical data (cf. subsection 1.3.3).

The second strategy of revision keeps the initial "more verities and fewer falsities" intuition behind the definition intact, but changes its interpretation in the Lindenbaum algebra. According to this revision the truth- and falsity-content of a theory consists only of the implied literals (representing empirical data).[45] This changes the fourth assumption at the end of subsection 1.2.5 (p. 13). Consequently, only the literals implied by a theory contribute to its distance to the truth. I shall abbreviate the new truth- and falsity-content of φ by

$$Ct'_T(\varphi) := \mathrm{Cn}(\varphi) \cap \{(\pm)p_i \mid (\pm)p_i \text{ is true}\} \text{ and}$$
$$Ct'_F(\varphi) := \mathrm{Cn}(\varphi) \cap \{(\pm)p_i \mid (\pm)p_i \text{ is false}\},$$

According to our reconstruction, the formal part of Popper's original definition remains the same.

DEFINITION 1.5: According to our second revision strategy, ψ is at least as close to the truth as φ iff
$Ct'_T(\varphi) \subseteq Ct'_T(\psi)$
$Ct'_F(\psi) \subseteq Ct'_F(\varphi)$
Notation: $\psi \leq^L_\tau \varphi$

If the literals of the algebra are connected with Popperian basic statements, then, clearly, according to the second way of revising Popper's definition, $\neg p \wedge \neg q$ is the worst theory of Prop($\mathscr{L}[p,q]$) if $p \wedge q$ is the truth. No other statement disagrees with the truth on more literals. Statement $\neg p \vee \neg q$ is more cautious; it does not make false claims about the literals. Consequently, according to this literal measure $\neg p \vee \neg q$ is much closer to the truth than $\neg p \wedge \neg q$. It is irrelevant that $\neg p \wedge \neg q$ has more true consequences than $\neg p \vee \neg q$; since only the literal consequences contribute to the distance to the truth. For the same reasons the literal measure favours $p \wedge \neg q$ to $\neg p \wedge \neg q$ if $p \wedge q$ is the truth. This rational reconstruction of the second revision of Popper's explanation foreshadows the likeness proposals as presented in Chapter 3. Of course, according to this second revision strategy the literals represent empirical basic statements and a theory is interpreted to be a large conjunction of literals (subsection 1.3.3).

Historically, likeness projects have emerged in different settings than the one just given. Martin Hyland, however, proposed a more general revision similar to our reconstruction,[46] he suggested representing theories by their bases. A basis of the theory φ is a set of independent axioms that, taken together, can produce all the sentences produced by φ; the notion comes from Tarski. The "more verities fewer falsities" intuition becomes "more true and fewer false elements in the basis." Miller produced an argument claiming that the resulting ordering depends on the arbitrary choice of the bases. Hyland's proposal is a more general approach than the one in which only the literals are considered. As Hyland's revision foreshadows is similar to other likeness proposals, unsurprisingly Miller articulates similar objections to other likeness proposals. We shall come to this objection in the last subsection 1.4.5. In Chapter 5, I present my solution to Miller's so-called "language dependency" objection.

Table 1. Revisions of Popper's Definition

	Definition	*Application* of the Lindenbaum Algebra
Content-revision	1. Formal definition: delete falsity clause 2. Truth-content more important than falsity-content	1. Intuitive interpretation w.r.t. the Lindenbaum Algebra: *all* logical consequences are equally important
Likeness-revision	1. Idea of the definition remains the same. 2. The truth- and falsity-content equally important	1. Intuitive interpretation w.r.t. the Lindenbaum Algebra: *only* literals count

To sum up, there are at least two ways to adjust Popper's original proposal such that it meets his requirements without giving up monotonicity. The first adapts the underlying "more verities fewer falsities" intuition of improvement, and preserves the interpretation in the Lindenbaum algebra; the second changes the interpretation of this algebra, and preserves the underlying improvement intuition. These two strategies result in two different sorts of orderings. The situation in which the Lindenbaum algebra is the mathematical application is sketched in Table 1.

1.4.2. Intuitive Characterization of Likeness and Content Definitions

In the preceding subsection, we interpreted the difference between content and likeness definitions as different revision strategies of Popper's original verisimilitude proposal. Here we depict the difference as it is articulated in the literature.

Content definitions order theories using their *logical content*, i.e. the extent of the set of consequences, or logical strength, and their *truth-values*. In other words, the sets of consequences of a proposition and its truth-value fix its verisimilitude. Consequently, according to a content definition, if ψ implies φ, and φ and ψ have the same truth value, then ψ is at least as verisimilar as φ. Furthermore, according to a content definition the verisimilitude of two sentences with the same truth-value and logical strength, is uncomparable. Both, Popper's original strategy and its revision, are examples of content definitions. Miller's definition and Kuipers's naive approach, both revisions of Popper's original proposal, are the content proposals presented in Chapter 2.[47] Oddie also uses the term *probability* definition, whereas Miller uses the term *algebraic* proposals; Kuipers calls them *naive* definitions.

Although Popper's intuitive characterizations of his verisimilitude idea uses the terms "likeness or similarity to truth", his formal proposal does not use the likeness notion.[48] The first likeness proposal was published in 1974. Tichý (1974) mentions a propositional example in which one false constituent is closer to the truth than another false constituent.[49] As the two constituents have the same logical strength, this is not a content ordering. Afterwards, Oddie formulated the following adequacy condition for likeness definitions: "equally strong theories may have different degrees of truthlikeness."[50]

Likeness definitions base their theory ordering on the *similarity* between the *constituents* (or *possible worlds)* of the language involved whereas logical strength only plays a subsidiary role. The idea is that within a conceptual framework, one (or more) model(s) modulo elementary equivalence completely represent the true theory τ; that is, all the consequences of τ are factually true, and within the conceptual framework there are no factually true statements that are no consequence of τ. If there is a similarity ordering of the models, the models of one theory can be more similar to that of τ than the models of another theory. Of course, the similarity of structures depends on the conceptual framework under consideration. Hilpinen was the first who explicitly called attention to the difference between content definitions such as that of Popper's and his own (qualitative) likeness definition.[51] The latter is based on a similarity relation between models, and is inspired by Lewis (1973). We will come to Hilpinen's proposal in Chapter 3. Besides the proposals of Hilpinen, and Tichý and Oddie, I will also discuss the likeness proposals of Niiniluoto, Kuipers (his refined proposal), Heidema Brink and Burger.

1.4.3. Formal Difference between Content and Likeness Ordering

Despite its significant role, I have not encountered a formal characterization of the content-likeness distinction. In the present subsection I fill this gap. First, we consider the minimal prerequisites of an approach-to-the-truth ordering. Let the empirical truth τ of a formal language $\mathscr{L}$ be complete. A successful approach-to-the-truth definition must at least be a *preordering* of EmProp($\mathscr{L}$), the empirical propositions of $\mathscr{L}$—i.e. the ordering must be transitive and reflexive. If for any ψ and φ $\psi \leq_\tau \varphi$ and $\varphi \leq_\tau \psi$ both apply, then ψ and φ have the same distance to the truth, and $\leq_\tau$ induces equivalence classes $[\varphi]^{\simeq}$ in EmProp($\mathscr{L}$), which is partially ordered by $\leq_\tau^{\simeq}$. Further, all propositions must be worse than τ; therefore, τ has to be the unique, *least element* of Prop($\mathscr{L}$). Regarding a preordering $\leq$, an element a is the least element of X if $a \in X$, and for all x $(\neq a) \in X$ $a \leq x$. The uniqueness requirement is necessary, as a preorder misses antisymmetry. All propositional verisimilitude orderings of Chapters 2–3, apart from a unique least element, also have a unique, *greatest element*. Regarding a preorder $\leq$, an element b is the unique, greatest element of X if for $b \in X$, and for all x $(\neq b) \in X$ holds $x \leq b$ and

$b \not\leq x$. Generally, an approach-to-the-truth ordering yields a set of *maximal* elements in Prop($\mathscr{L}$), max(Prop($\mathscr{L}$)) and need not to have a unique greatest element.

To simplify the general formal characterization of the differences between content and likeness definitions, I introduce the notion of the *complete falsehood* of a language as the "inverse" of the complete truth. To that end I first introduce the notions of the "inverse" of a literal l, $-(l)$, and the inverse of a conjunction of literals $-(\bigwedge_i l_i)$:

$$(15) \qquad -(l) := \begin{cases} \neg p & \text{if } l = p \\ p & \text{if } l = \neg p \end{cases}; \qquad -(\bigwedge_i l_i) := \bigwedge_i -(l_i)$$

DEFINITION 1.6: Let τ be the complete truth of a propositional language $\mathscr{L}$; then $\xi :=_{def} -(\tau)$ is the *complete falsehood* of $\mathscr{L}$

Regarding a finite propositional language, the complete falsehood contradicts the truth on every literal. It is a constituent.[52] Note that, as the completeness of the truth depends on the languages used, so does the completeness of its falsehood. Besides the negation of the truth the complete falsehood also implies numerous true propositions. Now that I have introduced the notion of a complete falsehood, I come to the formal criterium I use to distinguish content and likeness definitions.

DEFINITION 1.7: Let $\leq_\tau$ be an approach-to-the-truth ordering on EmProp($\mathscr{L}$), with least element τ and complete falsehood ξ. Then

a. $\leq_\tau$ is a *content* ordering if $\neg\tau$ is the greatest element of EmProp($\mathscr{L}$).

b. $\leq_\tau$ is a *likeness* ordering if the ξ is the greatest element of EmProp($\mathscr{L}$),

Obviously, according to a content ordering $\xi <_\tau \neg\tau$, and according to a likeness ordering $\neg\tau <_\tau \xi$. Suppose $\leq^C_\tau$ and $\leq^L_\tau$ designate a content and a likeness ordering, respectively. Then the following remark obtains:

Remark 6: If $\neg\tau \neq \xi$ (or, in other words τ is not a literal), then

1. $\forall\varphi \in$ EmProp($\mathscr{L}$): $\tau \leq^C_\tau \varphi \leq^C_\tau \neg\tau$
2. $\forall\varphi \in$ EmProp($\mathscr{L}$): $\tau \leq^L_\tau \varphi \leq^L_\tau \xi$
3. $\langle$EmProp($\mathscr{L}$), $\leq^C_\tau\rangle \neq \langle$EmProp($\mathscr{L}$), $\leq^L_\tau\rangle$

Obviously, the three clauses of the remark trivially follow from the definition. Thus, regarding $\mathscr{L}[p,q,r]$ with $\tau := p \wedge q \wedge r$, if $<^C_\tau$ and $<^L_\tau$ are a content and a likeness ordering of EmProp($\mathscr{L}[p,q,r]$), then the next inequalities represent the orderings of ξ and $\neg\tau$

$$(16) \qquad \neg p \wedge \neg q \wedge \neg r <^C_\tau \neg p \vee \neg q \vee \neg r$$
$$(17) \qquad \neg p \vee \neg q \vee \neg r <^L_\tau \neg p \wedge \neg q \wedge \neg r$$

In other words, according to likeness orderings in a finite propositional language with a complete empirical truth, the worst theory is the most distant constituent;

and the worst theory according to a content ordering is the negation of the truth. In the last Chapter, I develop a new proposal to merge the likeness and content approaches.

1.4.4. Child's Play and Language Dependency

Since the publications of Tichý and Miller in 1974, two kinds of objections have emerged time and again in approach-to-the-truth discussions. Tichý, as a supporter of the likeness approach, introduced the first objection, currently called the "child's-play objection."[53] He objects to the property of content definitions (def. 1.7) according to which every false antecedence of a false proposition φ is closer to the truth than φ itself.[54] This property makes it child's play to approximate the truth according to content definitions. Add a plainly false proposition χ—for instance, a fairy tale—to a falsified proposition φ, with $\varphi \nVdash \chi$ and $\chi \nVdash \varphi$, and the conjunction $\varphi \wedge \chi$ is closer to the truth than φ and χ. Hilpinen (1976) also mentions this criticism. Miller (1978) deplores this consequence of his own content proposal.[55] Note that the objection has an epistemological flavour. It claims that when compared with real, scientific endeavours, approaching the truth according to a content definition is much too easy; it suffices to add false propositions.[56]

The significance of Tichý's objection is qualified if the content definition receives a serious intuitive interpretation (subsection 1.3.3). In a propositional setting, the atomic sentences present highly abstract scientific hypotheses, which are only indirectly confirmable or falsifiable. Under these circumstances, adding a serious, abstract and independent hypothesis, which reckons with scientific background knowledge, might be a step in the right direction, although the hypothesis is likely to be false. Adding arbitrary false statements to an existing falsified theory does not result in a serious hypothesis.[57] Remember that, according to the content approach, only the extent of the set of true consequences fixes the verisimilitude of a hypothesis, and not its truth-value, as it is likely to be false.

The second constantly recurring objection comes from the ranks of the content proponents and is called the "language dependency" argument. It originates from Miller (1974) and instead of being epistemological, it is a formal argument. The argument runs thus: Apparently, $\neg p \wedge q$ is more like $p \wedge q$ than $\neg p \wedge \neg q$; however, if we 'translate' p into h, and q into $h \leftrightarrow g$, then $\neg p \wedge q \equiv \neg h \wedge (h \leftrightarrow g) \equiv \neg h \wedge \neg g$ is closer to $h \wedge g$ than $\neg p \wedge \neg q \equiv \neg h \wedge \neg(h \leftrightarrow g) \equiv \neg h \wedge g$.[58] I shall discuss this argument extensively in Chapter 5. For the moment, it is important to realize that a likeness interpretation of the propositions diminishes the impetus of the argument. The semantic relation between data and observational language is straightforward if compared with the semantics of the abstract hypotheses in a content definition. Consequently, it is questionable whether an extensional substitution from one language to another, preserves the original meanings. In Chapter 5, I shall show that this 'translation' represents a change of the relevant traits, and therefore a change of the cognitive problem. In practice, reformulating the empirical data will not cause any serious discussion about a change of the cognitive problem.[59] The problem is more complex if the paraphrase concerns abstract hypotheses.

In sum, supporters of likeness definitions reproach their content colleagues for defining verisimilitude such that it becomes a child's play to approach the truth; the content supporters, on their turn, blame their likeness companions for using definitions that are "language dependent." The importance of both objections decreases significantly when one takes the interpretation of the mathematical application seriously into account.

1.4.5. Approach-to-the-Truth; Verisimilitude and Truthlikeness

We have seen that the "more merits and fewer drawbacks" intuition has produced two kinds of formal explications. Content proposals base their ordering on truth-value and logical content, whereas according to likeness definitions the similarity between constituents (or possible worlds) fixes the truthlikeness of a theory. Consequently, all likeness proposals are "language dependent," whereas all content definitions are vulnerable to the child's-play objection. Moreover, the likeness interpretation of atomic sentences as basic statements qualifies the child's-play objection; and the content interpretation of atomic sentences as abstract hypotheses qualifies the "language dependency" argument. It is only at the end of Chapter 3 that I reveal the profound similarities between content proposals on the one hand, and of likeness explanations on the other. Nevertheless, here, I anticipate these conclusions, which are foreshadowed by the preceding arguments. Resembling the situation in which Carnap distinguished between probability$_1$ and probability$_2$, the approach-to-the-truth literature encompasses two different notions. I propose to use the term *verisimilitude* for "content proposals" and *truthlikeness* for the 'likeness proposals'. One might object that the former is the Latin translation of the latter, but I prefer a historically based terminology to a systematic one such as verisimilitude$_1$ and verisimilitude$_2$. It was Popper who introduced the technical term verisimilitude, and Hilpinen who stressed that his explanation was a truth*likeness* definition. Additionally, Niiniluoto's *Truthlikeness* has become a standard work in the field. Oddie called his contribution *Likeness to Truth*. I consider verisimilitude and truthlikeness both yield different contributions to the *approach-to-the-truth* project.

Let me summarize Section 1.4. Content and likeness orderings can be reconstructed as different strategies to revise Popper's original proposal. Beyond the intuitive description of the difference to be found in the literature, I gave a formal characterization. According to a content ordering, the worst theory is the negation of the truth; according to a likeness definition the complete falsehood is the worst theory. Additionally, content and likeness definitions must cope with the child's-play and language dependency objections, respectively.

1.5. MORE METATHEORETICAL PROPERTIES

In this section, I enumerate several important properties of approach-to-the-truth definition that will be used in the evaluations of those definitions in the chapters that are to come.

1.5.1. Strength of Definitions

In a general logical context, the statement "φ is stronger than ψ" means $\varphi \vDash \psi$ and $\varphi \not\equiv \psi$, that is $Cn(\varphi) \supset Cn(\psi)$. If I use this expression to compare relations, it is likely to cause misunderstandings since

(18) "relation $\leq$ is stronger than the relation $\leq'$ "

is ambiguous. It is unclear whether (18) is about the *meta-level* or the *object-level* of the relations. I shall explain the situation using the difference of strength between the modal systems S4 and S5. It is common to call S5 a stronger system than S4; the consequences of the S5 axioms contain those of S4 axioms. Schematically

$$S5 \Rightarrow S4, \text{ or } Cn(S5\text{-}Axioms) \supset Cn(S4\text{-Axioms})$$

I shall call this the *meta-level reading* of (18) as entire modal systems are compared. Hence, on the *object-level*

$$\forall \varphi, \psi{:}\ \varphi \vdash_{S4} \psi \rightarrow \varphi \vdash_{S5} \psi,$$

and all S4 consequences of a sentence φ, are also S5 consequences of φ. I call this the object-level reading of (18), since the individual sentences are compared. The same situation obtains for relations in general. For instance, on the meta-level, we have

If R is a partial ordering, then R is a preordering

If, however, a property $C(x,y)$ follows from the fact that x and y are in the R_{pre} relation, then $C(x,y)$ follows a fortiori from the fact that x and y are in the $R_{partial}$ relation.

$$\forall R_{pre} [\forall x,y\, (R_{pre}(x,y) \rightarrow C(x,y))] \Rightarrow \\ \forall R_{partial} [\forall x,y\, (R_{partial}(x,y) \rightarrow C(x,y))]$$

The difference between the meta- and object-level reading of (18) is common in logic, but it has caused some discussions in the approach-to-the-truth project. Using the object-level explanation, some call the ordering $\leq$ stronger than $\leq'$ if

$$\text{for all } \varphi, \psi\text{:}\ \ \varphi \leq \psi \ \Rightarrow\ \varphi \leq' \psi$$

In other words, if the $\leq$ pairs of sentences must fulfill stronger constraints than the $\leq'$ pairs, then the former is the stronger ordering. Others, using the meta interpretation, have called $\leq$ weaker in exactly the same situation, since the $\leq'$ axioms imply the $\leq$ axioms or:

$$\mathrm{Cn}(\leq\text{-}Axioms) \subseteq \mathrm{Cn}(\leq'\text{-}Axioms).^{60}$$

This reading implies that some approach-to-the-truth definition is stronger than another one, if $\leq'$ relates more pairs of sentences than $\leq$. To preclude any misunderstandings I explicitly formulate my choice:

Convention: I use the expression "the $\leq'$ ordering of propositions is stronger than the $\leq$ ordering" in the *meta-level* sense—the way in which S5 is stronger than S4. Consequently, the *stronger* ordering compares *more* pairs of sentences.

According to this meta-level interpretation, the set $\leq'$ *Axioms* is stronger than the set of $\leq$ *Axioms* iff $\leq'$ *Axioms* $\vdash$ $\leq$ Axioms. This choice implies on the object level that all $\leq$ pairs are also $\leq'$ pairs:

$$\mathrm{Cn}(\leq\text{-}Axioms\ ) \subseteq \mathrm{Cn}(\leq'\text{-}Axioms).$$

This implies that for all φ, ψ if they are comparable by $\leq$, then they are comparable by the $\leq'$ relation. It is the latter circumstance that persuaded some authors to claim the opposite, viz. that in this situation $\leq$ is stronger than $\leq'$. There is nothing utterly wrong with that choice, but when calling a relation strong (or weak), one is obliged to be more explicit about the meaning of that expression.

In sum, according to the *object-level* interpretation, the $\leq$ ordering is *stronger* than $\leq'$, iff $\forall\varphi, \psi$: $\varphi \leq \psi \Rightarrow \varphi \leq' \psi$. According to the *meta-level* interpretation, which I shall use in the sequel, in the same circumstances, $\leq$ is *weaker* than $\leq'$, since $\leq'$ handles more pairs of sentences than $\leq$; and hence $\mathrm{Cn}(\leq\text{-}Axioms\ ) \subseteq \mathrm{Cn}(\leq'\text{-}Axioms)$.

1.5.2. Comparative and Quantitative Definitions

Approach-to-the-truth definitions have been classified in various ways, and I shall show that the content-likeness distinction is by far the most important one. Another important distinction, however, is the comparative-quantitative distinction. Characteristically comparative definitions claim that one theory is more (or at least as) similar to the truth than a second theory. It is a comparative claim. Quantitative definitions, however, assign numbers to theories conveying to what extent a theory resembles the truth. This might appear artificial at first sight, but in the natural sciences most claims are quantitative; and comparing sets of quantitative assertions easily results in a quantitative assessment. Quantitative definitions often yield reflexive, and therefore preorderings, and comparative proposals may be irreflexive resulting in strict partial orderings.

The content-likeness distinction pertains to quantitative and to qualitative definitions. In other words, there are quantitative and qualitative *content* definitions—Popper elaborated both—and there are quantitative and qualitative *likeness* proposals such as those of Tichý or Niiniluoto at the one hand, and of Hilpinen at the other. The vast majority of the mature sciences use quantitative hypotheses and theories. A quantitative definition enormously simplifies the comparison of these quantitative statements. Another advantage of using a quantitative truthlikeness definition is the increase of strength of comparison. Generally, qualitative theory comparison establishes *partial orderings* (or even preorderings) among the formal statements; this means that using the definition we cannot compare all sentences. A quantitative explanation however, gives a *total* ordering, and therefore all theories can be mutually compared. An important question about a quantitative proposal is, whether it provides a *metric* or some weaker *distance function* on the class of theories. I shall deal with the *metric*, *semimetric*, and *pseudometric*. First, we consider the definition of a metric:

DEFINITION 1.8: Let X be some universe, then $d\colon X \times X \to \mathbb{R}$ is a *metric* on X iff for all $x, y, z \in X$ hold

1. $d(x, y) \geq 0$
2. $d(x, y) = 0$ iff $x = y$
3. $d(x, y) = d(y, x)$ (symmetry condition)
4. $d(x, y) + d(y, z) \geq d(x, z)$ (the triangle inequality)

Notation: (X, d) is called a *metric space.*

E.g. the Euclidean metric on $\mathbb{R}^n$ is:[61] $d(x,y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}, \quad n \in \mathbb{N}$

Instead of working with a metric, mathematicians also use weaker notions. If clause 2. of definition 1.8 is weakened to $d(x, x) = 0$, then d is a *pseudometric* on X. The second weaker alternative for the metric is the *semimetric*. Distance d is a (symmetric) semimetric if it complies with the first two (three) clauses of definition 1.8. Niiniluoto remarks "In many cases, symmetric semimetrics are called *distance functions* even if they do not satisfy the triangle inequality."[62] Thus, Niiniluoto's favourite distance function is not a metric. Miller maintains, however, that a distance function must apply to the triangle inequality.[63] Kuipers's refined quantitative distance function is also an example of a symmetric semimetric.

A disadvantage of using a quantitative proposal is the rather arbitrary choice of the measure. There are many measures that can lead to an approach-to-the-truth definition avoiding the mistake of Popper's original proposal;[64] but, if only intuitive arguments lead to the preference of some particular metric, the verisimilitude notion becomes too *subjective*. This is an important reason for first establishing a compelling, qualitative truthlikeness definition and subsequently extending the proposal towards a stronger one, which might ultimately result in an objective quantitative notion.

Finally, I want to express my reservations about treating quantitative and qualitative expressions on the same footing. In subsection 1.3.2 we saw that many definitions have quantitative *and* qualitative applications. In the subsection that followed however, we stressed the importance of the interpretations of the applications. Let for instance $p_9, \ldots, p_{19}$ represent the atomic statements "Our sun has exactly nine planets", until "Our sun has exactly nineteen planets." Then, p_{19} may be the worst possible answer to the question "How many planets does the sun have?", and all the consequences of these atomic sentences are totally left out of the considerations. If, on the other hand, $p_9, \ldots, p_{19}$ represent highly abstract theoretical laws, then the extent of the true and false consequences seems to be more important than the distances between the individual constituents. The conclusion reads that the similar treatment of quantitative and qualitative applications apparently needs good arguments.

1.5.3. Truth-value Independence

Quite surprisingly, until now, the subject of this subsection has received almost no attention. Intuitively, it seems plausible that, if two theories are roughly of the same logical strength, then a true theory is to be preferred to a false one. There are situations, however, where a false theory is by far to be preferred to a true theory. If the true theory is genuinely weak (it is almost a tautology), then a strong, relevant, and almost true theory is certainly better than the very weak true theory. In other words, intuitively, a *false* theory sometimes is closer to the truth a *true* one.[65] Let $\leq_\tau$ be an approach-to-the-truth ordering; then, I call an approach-to-the-truth definition *truth-value dependent* iff no false theory is closer to the truth than a true one:

$$\forall \varphi, \psi \in \mathrm{Prop}(\mathscr{L})\text{: if } \tau \vDash \psi \text{ and } \varphi \vDash \neg\tau, \text{ then } \varphi \nleq_\tau \psi$$

An ordering is truth-value independent if it is not truth-value dependent. This property appears to be weak, but it abolishes the suppression of false propositions. It is worth mentioning because *all major qualitative*, content and likeness orderings are *truth-value dependent*. That is, according to all qualitative proposals available, no false sentence is closer to the truth than any true one. Of course, this is quite a shortcoming. My proposal given in Chapter 6 is the first published truth-value *in*dependent *comparative* approach-to-the-truth definition.

1.5.4. Specularity

The fourth property of an approach-to-the-truth ordering $<_\tau$ I consider is the *specularity*. Let τ be the complete truth of $\mathscr{L}$. Then, if for all ψ and φ of Prop($\mathscr{L}$)

$$\psi <_\tau \varphi \Leftrightarrow \neg\psi <_{\neg\tau} \neg\varphi$$

obtains, then the $<_\tau$-ordering has the specularity property.[66] To begin with, I show that no likeness ordering has the specularity property.

PROPOSITION 1.7: A strict truthlikeness ordering $<^L_\tau$ does not have the specularity property.

Proof: If $<^L_\tau$ is a strict likeness ordering, then for all $\varphi \in$ Prop($\mathscr{L}$) $\tau <^L_\tau \varphi <^L_\tau \xi$ obtains with $\xi \not\equiv \neg\tau$, and $\xi \vDash \neg\tau$ (see subsection 1.4.3). If specularity obtained, this would lead to a contradiction since then, $\neg\tau <^L_{\neg\tau} \neg\varphi <^L_{\neg\tau} \neg\xi$, and $\neg\tau \vDash \xi$ obtained, and this implies $\neg\tau \equiv \xi$. ⊠

Further, since most important content proposals we encounter have the specularity property, we consider specularity to be an indication of content definitions.

1.5.5. Language Dynamics

The approach-to-the-truth debate also touches upon the question how definitions behave under expansion and reduction of the formal language (including theoretical terms).[67] We call this the *language-dynamic* behaviour of the definitions. Language $\mathscr{L}'$ is an *expansion* of $\mathscr{L}$ and $\mathscr{L}$ a *reduction* of $\mathscr{L}'$, $\mathscr{L} \subseteq \mathscr{L}'$, if the nonlogical vocabulary of $\mathscr{L}$ is a subset of that of $\mathscr{L}'$.[68] Regarding a propositional language, $\mathscr{L}'$ is an expansion of $\mathscr{L}$ if all atomic propositions of $\mathscr{L}$ are in the nonlogical vocabulary of $\mathscr{L}'$: voc($\mathscr{L}$) $\subseteq$ voc($\mathscr{L}'$). Suppose $\mathscr{L}$ is a formal language, with a (complete) empirical truth τ. Let, for all $\mathscr{L}'$ with $\mathscr{L} \subseteq \mathscr{L}'$ and complete truth τ', ($\tau' \vDash \tau$), $<_\tau$ and $<_{\tau'}$ be the orderings of Prop($\mathscr{L}$) and Prop($\mathscr{L}'$), respectively; and let $\sigma(\varphi)$ be the $\mathscr{L}'$-translation of φ—i.e. $\sigma(\varphi) \equiv_{\mathscr{L}'} \varphi \in$ Prop($\mathscr{L}$).

DEFINITION 1.9: Ordering < is *(weakly) context independent* iff
$\forall\psi, \varphi \in$ Prop($\mathscr{L}$), and $\sigma(\psi), \sigma(\varphi) \in$ Prop($\mathscr{L}'$): $\psi <_\tau \varphi \Rightarrow \neg(\sigma(\varphi) <_{\tau'} \sigma(\psi))$;
it is *strongly context independent* iff
$\forall\psi, \varphi \in$ Prop($\mathscr{L}$), and $\sigma(\psi), \sigma(\varphi) \in$ Prop($\mathscr{L}'$): $\psi <_\tau \varphi \Rightarrow \sigma(\psi) <_{\tau'} \sigma(\varphi)$.

Intuitively, an ordering is context independent if for no pair of propositions an expansion of the language reverses the ordering of that pair. It is strongly context independent if for all pairs of sentences the ordering in the original language is preserved in any expansion of the language. In Chapter 2 I show that some content proposals fail to meet the latter requirement. Note that what we call "language dynamics" of approach-to-the-truth definitions has little to do with recent developments in "logical dynamics". In the latter the logic of changing information states and semantics in linguistics is studied.[69]

Alongside the expansion of the empirical vocabulary, the extension of the logical vocabulary is also part of the language dynamic behaviour of approach-to-the-truth definitions. For example, questions as to how do orderings behave if the logical machinery of the language is extended or changed, need to be addressed. Apart from some occasional remarks, the study of this dynamic behaviour is beyond the scope of our investigation. We call an approach-to-the-truth ordering *logically biassed* if it fundamentally depends on the logical machinery of the language.

1.5.6. (In)complete Truth

In Chapter 2–3 we shall see that most researchers assume that the truth is complete. Kuipers is one of the few who departs from this well-trodden path. Using the structuralist approach he suggests that the true theory must be paraphrased by a set of models instead of one model (modulo elementary equivalence). Consequently, Kuipers pursues a definition of what he calls "theoretical truthlikeness."[70] At the start, this difference of assumption complicates the mutual comparison of the proposals. Niiniluoto, however, gives two suggestions to bring Kuipers's point of view on a par with the other definitions, and leaves the details to the reader.[71] In Chapter 2, I elaborate one of these suggestions. After all, it comes out that a modal version of Kuipers's approach also assumes a complete truth. Under those circumstances, his naive definition is equivalent to Miller's content proposal. Incomplete truth is a symptom of a semantically indeterminate or, in Niiniluoto's terms, indefinite conceptual framework.[72]

DEFINITION 1.10: Let $\mathscr{L}$ be a formal language and $\mathscr{I}$ be the empirical interpretation of $\mathscr{L}$ ($\mathrm{voc}(\mathscr{L}) \neq \varnothing$). The interpreted language $\langle \mathscr{L}, \mathscr{I} \rangle$ is *semantically definite* if all sentences of $\mathscr{L}$ are true or false.

The definition implies that $\mathscr{L}$ is semantically definite if the empirical true theory of $\mathscr{L}$ is complete; the language is indefinite if some sentences lack a truth-value.

The completeness of the truth is intimately connected with the language dynamic properties of a proposal. Regarding approach-to-the-truth definitions, we may assume the completeness of the truth. In epistemological contexts, however, this assumption is unwarranted. The language in which the theories are formulated may include non-referring terms, and therefore contain sentences without a definite truth-value. Suppose we are to assess a real-life scientific theory φ formulated in the language $\mathscr{L}_\varphi$. Further, I assume that $\mathscr{L}_\varphi$ contains theoretical terms relative to theory φ. It follows that $\mathscr{L}_\varphi$ can only be partially interpreted, and it is uncertain whether all its sentences are false or true. For instance, there were times in which it was unclear whether the statement

"phlogiston has a negative weight"

had a definite truth-value. According to our knowledge, the statement contains a non-referring term, and it is neither false nor true, but senseless. As long as it is unclear whether the theoretical terms refer to some aspects of reality, some sentences may lack a truth-value. This is the reason for rejecting the assumption that the truth is complete in epistemological contexts. The task remains to investigate the behaviour of the definitions under indefinite language extension. Thus, a relevant problem reads: "how do the definitions behave if the language is indefinitely extended?" To answer this question, an approach-to-the-truth proposal must also order propositions in case some relevant sentences do not receive a definite truth-value. To sum up, when defining verisimilitude, we may assume that the truth is complete. This assumption, however, is only allowed as a first idealization; in the epistemological context, the assumption is unwarranted.

1.6. CONCLUSIONS

Let me enumerate the highlights of Chapter 1.

1. According to Popper's original verisimilitude definition a theory ψ is at least as close to a theory φ iff $Cn^F(\psi) \subseteq Cn^F(\varphi)$ and $Cn^T(\psi) \supseteq Cn^T(\varphi)$. Consequently, two different false ψ and φ are uncomparable, because under the assumption of a monotone language, the non-empty subset of *false consequences* of a proposition *varies* with the subset of its *true consequences*.
2. My diagnosis of Popper's mistake reads: Popper used *monotonic means* to formalize his *non-monotonic* approach-to-the-truth *intuitions*.
3. I will use *propositional languages,* in fact *finite Lindenbaum algebras,* to compare various proposals that improve Popper's definition.
4. In the chapters that are to come, we will see that the paramount division in the approach-to-the-truth research is that between *content and likeness* definitions.
5. Finally, the *truth-value dependency*, the *child's play, invariance under extensional substitutions*, and the *language dynamic behaviour* in general are the important properties of the definitions studied in the following chapters.

CHAPTER 2

VERISIMILITUDE

In Chapter 1, I introduced Popper's comparative content definition, which is based on truth-value and logical strength, and showed how it excludes the comparison of two different false theories. After the publication of this peculiarity in 1974,[1] Miller and Kuipers searched independently for another way to formalize Popper's intuitions about verisimilitude. Their endeavours resulted in *comparative* content definitions (Kuipers calls his version the *naive* comparative definition). Miller formulates a distance function which has as its codomain the original Boolean algebra instead of the real numbers. Both definitions are almost identical to the consequence definition also hinted at in the first chapter (subsection 1.4.1). Here we examine Miller's and Kuipers's content proposals.

In the present and next chapter, I shall present the content and likeness orderings of the elements of a finite Lindenbaum algebra. In Chapter 1, we saw that the definitions vary considerably regarding their applications and interpretations, and that each has its own level of abstraction. This complicates the comparison. Without exception, however, the definitions can all be used to order propositions, and therefore, the Lindenbaum algebra provides good grounds for a thorough basic comparison. There are also other reasons for choosing a Boolean algebra as the basic application for the comparison. The Boolean algebra relates to the algebras of sets such as the *Brouwerian* algebra and the *Stone space*, which form more sophisticated interpretations. Another important extension concerns modal algebras. We shall encounter all of these in due course. As it will turn out, already with respect to Boolean Algebras, the various definitions will produce different orderings.

In Section 2.1, I introduce the notion of a Lindenbaum algebra which is central to my comparison. Miller's (1978) definition, and Kuipers's comparative content definition are introduced in the second and third section. Their proposals are very similar, and both turn out to be weaker versions of the consequence definition. In Section 2.4, we discuss various ways to formulate the symmetric difference measure. The characteristics of Miller's and Kuipers's proposals are the subject of Section 2.5; and finally, in Section 2.6, I reformulate Kuipers's content definition in modal terms, and end with a modal version of the consequence definition that fully restores the falsity clause.

2.1. THE LINDENBAUM ALGEBRA

Let us consider a language $\mathcal{L}$ which has as a vocabulary, voc($\mathcal{L}$), a (possibly infinite) set of atomic propositions. As explained in Chapter 1, a Lindenbaum (-Tarski) algebra $\mathcal{B}$ of $\mathcal{L}$, is a Boolean algebra, the elements of which are equivalence classes of $\mathcal{L}$ sentences. For all practical purposes, we let one element of such an equivalence class represent the entire class, such that the elements of the algebra are single sentences. In this Boolean algebra, the 0 and 1 represent the contradiction and the tautology, respectively. Furthermore, the usual + and . represent the disjunction and the conjunction of $\mathcal{L}$; and the $\leq$ sign refers to derivability in $\mathcal{L}$. An element t is an *atom* in the Lindenbaum algebra $\mathcal{B}$ iff $a = 0$ obtains for all a such that $a < t$. A theory is complete if it renders all sentences of the language under consideration false or true. Thus, t is an atom in the algebra if and only if t is complete. Often, a Boolean algebra is isomorphic to an algebra of sets. Then, the $\leq$ sign represents the set inclusion, and the + and the . in $\mathcal{B}$ (the disjunction and conjunction of $\mathcal{L}$) represent the union and intersection of the set algebra, respectively.

EXAMPLE: As an example we sketch the Lindenbaum algebra, $\mathcal{B}$ of $\mathcal{L}[p, q]$. This $\mathcal{L}$ is a propositional language with descriptive vocabulary $\{p, q\}$. Recall that the elements of $\mathcal{B}$ are sets of equivalent sentences. In our representation, we let single sentences represent those equivalence classes; and the lower of two connected sentences implies the upper one. For instance $p \wedge q$ implies $\neg p \vee q$.

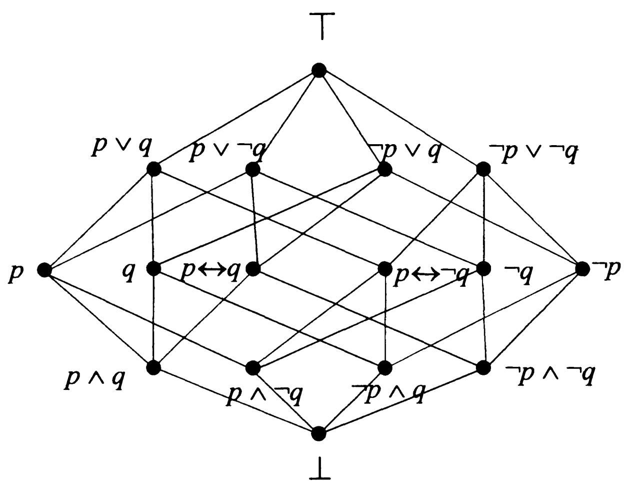


Fig. 1. Lindenbaum algebra $\mathcal{B}$ of $\mathcal{L}[p,q]$.

All the elements of the second layer are atoms of the algebra. They are the constituents of $\mathscr{L}[p,q]$. Of course, if voc($\mathscr{L}$) is finite, every theory of $\mathscr{L}$ is axiomatizable; however, this is different if voc($\mathscr{L}$) is infinite. Intuitively, this means that the atomic propositions p_i which are true according to the theory, can form such a haphazard set that it is impossible to enumerate all of them by any recursive method. With respect to a propositional language with an infinite vocabulary $\mathscr{L}[p_1, p_2, \dots]$, the elements of the Lindenbaum algebra only represent the axiomatizable theories. The infinite Lindenbaum algebra does not contain atoms since for all elements a in the algebra, there is an element b with $0 < b < a$. Thus, if the truth of $\mathscr{L}[p_1, p_2, \dots]$ is complete, then it is not an atom of the algebra, although it might be axiomatizable. *End Example*

2.2. MILLER'S CONTENT DEFINITION

Miller's verisimilitude definition is to be found in Miller (1978). Its main message is that on a Boolean (set) algebra, the symmetric difference is the only *normal, downward strictly monotone autometric* (I shall explain these notions below), and therefore it is the most appropriate distance measure on the algebra. It establishes a verisimilitude ordering as the distance between theories is inversely proportional to their similarity.

Miller's (1978) publication consists of three parts. The first one concerns the symmetric difference measure on a *Lindenbaum* algebra (the first and second section of Miller (1978)), and the second, an autometric on *Brouwerian* algebras, which represent richer languages than the propositional ones (his third section). Unfortunately, in contrast to Lindenbaum algebras, there is no autometric operation on a Brouwerian algebra that is normal and downward strictly monotone. Then, in the third part (the fourth section), Miller proposes a second strategy to extend his Lindenbaum approach. This model theoretical approach uses the autometric on the Boolean set algebra of the *Stone space* of the $\mathscr{L}$-models modulo elementary equivalence. Miller shows that "this representation effectively allows us to forget about Brouwerian algebras altogether."[2] The important results of his Brouwerian strategy, however, remain valid. Although I sketch the three parts of the paper, I shall take the model theoretic version of Miller's proposal as the outcome of his considerations (the impatient reader may skip the sequel and pick up the thread at definition 2.1. (page 42).[3]

2.2.1. The Symmetric Difference on the Lindenbaum Algebra

Miller's point of departure is to take literally the idea that theories "can be close to or distant from one another."[4] This distance function, which applies to theories, has to fulfill the customary axioms of a metric. Such a distance function $d(B,A)$ for theories must satisfy the customary conditions (subsection 1.5.2):

$$d(A, B) = 0 \text{ if and only if } A = B$$
$$d(A, B) = d(B, A)$$
$$d(A, C) \leq d(A, B) + d(B, C) \text{ (the triangle inequality)}$$

Obviously, just as in the Euclidian space, there are lots of distance functions fulfilling these axioms. How can the number of distance functions be reduced? This can be achieved, by choosing the original Boolean algebra as the codomain for the function instead of the real numbers. Miller uses Ellis's (1951) claim that the *symmetric difference* of a and b, defined by

$$\Delta(a, b) := (a.-b) + (b.-a)$$

fulfils the customary axioms of a distance function. Note that in a Boolean set algebra $\Delta(a, b)$ designates $(a-b) \cup (b-a)$ and in a Lindenbaum algebra, it means $a \leftrightarrow \neg b$. According to Ellis, a Boolean algebra $\mathcal{B}$ is *autometrized* if the codomain of its metric is $\mathcal{B}$ itself. The symmetric difference fulfils this demand. Thus, Miller calls the distance function * on a Boolean algebra $\mathcal{B}$ with codomain $\mathcal{B}$ an *autometric*.

The symmetric difference is not the only possible autometric. Miller introduces two conditions that together are necessary and sufficient for an autometric to be equal to the symmetric difference. Following Ellis, he calls an autometric *normal* if there is an element e such that for all $a \in \mathcal{B}$: $a * e = a$. In a Lindenbaum algebra e is the contradiction. Next, he calls an autometric operation *downward strictly monotone* iff

$$c < b < a \Rightarrow b{*}a < c{*}a$$

Miller proves subsequently that if * is a normal, downward strictly monotone autometric on the Boolean algebra $\mathcal{B}$, it is the symmetric difference. If being normal and downward strict monotonicity are necessary conditions for a verisimilitude definition, then Miller has found objective reasons to sort out the symmetric difference as the favourite metric on a Boolean algebra. Let τ be the complete truth of a propositional language; in other words, τ is an *atom* in the Lindenbaum algebra $\mathcal{B}$ ($\mathcal{B}$ is finite). The theory ψ is more verisimilar than φ if its distance to the truth is smaller than the distance between φ and the truth τ. In terms of the symmetric difference on $\mathcal{B}$ this definition reads:

(1) $\psi \Delta \tau \leq \varphi \Delta \tau$ (that is $\psi \leftrightarrow \neg\tau \vDash \varphi \leftrightarrow \neg\tau$, or $\varphi \leftrightarrow \tau \vDash \psi \leftrightarrow \tau$; see Observation 2.2, p. 52)

If we assume, following Miller, the completeness of the truth, this definition has three consequences of which the first two are also consequences of Popper's definition:[5]

(2) "the truth content of a false theory is closer to the truth than is the false theory itself," since the truth content of a false theory φ is equal to $\varphi \vee \tau$, and $\varphi \leftrightarrow \tau \ \vDash \ (\varphi \vee \tau) \leftrightarrow \tau$

(3) "the stronger of two comparable true theories is closer to the truth," since if $\tau \vDash \psi$ and $\tau \vDash \varphi$ (†), then $\psi \leftrightarrow \neg\tau$ equals $\psi \wedge \neg\tau$. Therefore, definition (2.1) reduces to $\psi \wedge \neg\tau \vDash \varphi$, which with (†), equals $\psi \vDash \varphi$

(4) "the stronger of two comparable false theories is closer to the truth", since if $\psi \vDash \neg\tau$, then $\psi \leftrightarrow \neg\tau \equiv \psi \vee \tau$. Thus definition (2.1) reduces to $\psi \vDash \varphi \vee \tau$; but as $\psi \vDash \tau$ it follows that $\psi \vDash \varphi$ (*the child's-play objection*).

It is the last conclusion that Miller finds "altogether less amusing". After all, it means that for false theories, and all actual theories are strictly speaking false, the verisimilitude relation between theories is identical to the relation of logical deduction.

In the next two subsections, we shall encounter Miller's endeavour to extend his autometric proposal for propositions, $\varphi \leftrightarrow \tau \vDash \psi \leftrightarrow \tau$, to the case in which the truth in not finitely axiomatizable. First, he seeks an adequate autometric on the Brouwerian algebra in which every element is a consequence class. After the failure of this attempt, he successfully considers the autometric on Stone space of the models of $\mathscr{L}$. First, I introduce Miller's first attempt; next I shall present the autometric on the Stone space.

2.2.2. No Symmetric Difference Measure on Brouwerian Algebra's

In the second part (section three) of Miller's paper, the answer to the question how theories should be ordered when couched in richer than finite propositional languages is formulated. Since, then, the set of all true propositions need not be finitely axiomatizable, Miller uses Tarski's *calculus of deductive systems*. A deductive system A is a set of sentences closed under logical deduction, $A = \mathrm{Cn}(A)$. In Tarski's calculus, the set inclusion partially orders the deductive systems, which form a lattice. Consequently, the greatest lower bound of A and B is $A \cap B$. The least upper bound is, contrary to one's first intuitions, not equal to $A \cup B$ since $\mathrm{Cn}(A) \cup \mathrm{Cn}(B) \neq \mathrm{Cn}(A \cup B)$. For instance, if $A := \mathrm{Cn}(\varphi \rightarrow \psi)$ and $B := \mathrm{Cn}(\varphi)$, then $\psi \notin \mathrm{Cn}(A) \cup \mathrm{Cn}(B)$ but $\psi \in \mathrm{Cn}(A \cup B)$. The least upper bound is defined by $\mathrm{Cn}\{a \wedge b \mid a \in A, b \in B\}$. The top of the algebra is equal to Sent($\mathscr{L}$), the set of all $\mathscr{L}$-sentences, and the bottom is the set of tautologies. Miller chooses to use the dual of Tarski's calculus to simplify the comparison with the Lindenbaum algebra top and bottom of which are the tautology and the contradiction, respectively. It is the lattice in which $\leq_{\mathscr{A}}$ refers to the $\supseteq$-relation instead of $\subseteq$-relation. In this *Brouwerian algebra* $\mathscr{A}$, the lowest upper bound of A and B, $A \vee_{\mathscr{A}} B$, is $A \cap B$, whereas their greatest lower bound, $A \wedge_{\mathscr{A}} B$, is $\mathrm{Cn}\{a \wedge b \mid a \in A, b \in B\}$.[6] The top of the lattice, $\top_{\mathscr{A}}$, is the set of tautologies, and the bottom, $\bot_{\mathscr{A}}$, is Sent($\mathscr{L}$). In the Brouwerian algebra, nothing corresponds to the implication, and the strongest system C, such that $B \leq_{\mathscr{A}} A \vee_{\mathscr{A}} C$, defines the *difference* $B -_{\mathscr{A}} A$: $B -_{\mathscr{A}} A \leq_{\mathscr{A}} C$ iff $B \leq_{\mathscr{A}} A \vee_{\mathscr{A}} C$. The symmetric difference between systems A and B is equal to $(B -_{\mathscr{A}} A) \vee_{\mathscr{A}} (A -_{\mathscr{A}} B)$.[7] Finally, the system contrary to B, $\neg_{\mathscr{A}} B$, is the largest system X such that $B \vee_{\mathscr{A}} X = \top_{\mathscr{A}}$.[8] Thus $\neg_{\mathscr{A}} B$ is the largest consequence class of $\mathscr{A}$ that only shares the tautologies with B.

Analogous to the propositional case, Miller calls a function * an *autometric* iff

$$A * B = \bot_{\mathscr{A}}, \text{ iff } A = B$$
$$A * B = B * A$$
$$A * C \leq (A * B) \vee_{\mathscr{A}} (B * C)$$

and again, an autometric * is normal if there is a $\bot_{\mathscr{A}}$, such that for all A, $A * \bot_{\mathscr{A}} = A$. Adapting a proof of Nordhaus and Lapidus (1954), Miller shows that the symmetric difference defined above is an autometric operation. Further, he proves that if * is a normal autometric on $\mathscr{A}$, then $A \,\Delta_{\mathscr{A}}\, B \leq A * B \leq A \vee_{\mathscr{A}} B$. The major difference with the propositional case, however, is that Miller also proves that there is *no* autometric operation on $\mathscr{A}$ which is normal *and* downward, strictly monotone.

Regarding the truth T, Miller assumes that it is complete but *not finitely axiomatizable* (T is a set of sentences). This non-axiomatizability assumption has far-reaching consequences. In the first place, Miller proves that for all complete X that are not axiomatizable, the complement of X in the Brouwerian algebra, $\neg_{\mathscr{A}} X$, equals the set of all tautologies.[9] For, let y be a non-tautological consequence of $\neg_{\mathscr{A}} X$, and let y not be a consequence of X. Then, for all $x \in \mathrm{Cn}(X)$, $y \vee x \in \mathrm{Cn}(X)$. Thus, for any $x \in X$, $\neg_{\mathscr{A}} y \vDash x$ and X is, contrary to our assumption, axiomatizable; in one word, if X is complete and not axiomatizable, then $\neg_{\mathscr{A}} X = \top_{\mathscr{A}}$. Secondly, Miller proves that under his truth assumptions, for all axiomatizable A, $A * T = A \vee T$; in words: "*The distance from the truth of an axiomatizable theory is equal to its truth content*" or $(B -_{\mathscr{A}} T) \vee_{\mathscr{A}} (T -_{\mathscr{A}} B)$ reduces to $B \vee_{\mathscr{A}} T$.[10] Furthermore, Miller shows that if A and B are axiomatizable, then $B \vee T \vdash A \vee T$ iff $B \vdash A$.[11] The only possible exception would be when A is false, B is true, and $B = A \vee_{\mathscr{A}} T$; but then B cannot be axiomatizable since no truth content of a false theory is axiomatizable when the truth is not axiomatizable.[12] Thus, the exception of the finite case disappears if the language is infinitely large. Miller's assumption that the truth is not axiomatizable, leads to the following conclusion: if A and B are axiomatizable, then

(5) $B * T < A * T$ if and only if $B < A$. (*Miller's objection*)

That is to say: If A and B are axiomatizable, then B is closer to the truth T (not axiomatizable) if and only if A is a subset of B; but A being a subset of B is not only necessary and sufficient for B to be closer to the truth; it is also necessary and sufficient for B to be closer to every *arbitrary*, complete theory X that is not axiomatizable. The last observation leads to the conclusion that "approach to the truth via axiomatizable theories is quite *independent* of where the truth in truth is."[13] In the fourth subsection, I will show yet another remarkable consequence of the assumption that the truth is not axiomatizable.

The conclusion of this subsection reads: The symmetric difference $\varphi \leftrightarrow \tau \vDash \psi \leftrightarrow \tau$, is an appropriate comparative measure on axiomatizable theories, though it fails to be an appropriate autometric for consequence classes. This need not surprise us; Popper's original definition (p. 8) already failed for the axiomatizable case.

2.2.3. *The Stone Space Ordering*

After the failure of the Brouwerian algebra, Miller extends, in the fourth section, the Lindenbaum algebra approach into a model theoretic direction. We saw (p. 38) that $\psi \,\Delta\, \tau \leq_{\mathcal{B}} \varphi \,\Delta\, \tau$ designates $\psi \leftrightarrow \neg\tau \vDash \varphi \leftrightarrow \neg\tau$, which, in the Lindenbaum algebra means $\mathrm{Mod}(\psi) \,\Delta\, \mathrm{Mod}(\tau) \subseteq \mathrm{Mod}(\varphi) \,\Delta\, \mathrm{Mod}(\tau)$.[14] This enables Miller to consider the *Stone space* of the Lindenbaum algebra $\mathcal{B}$ if the truth is not axiomatizable (A, B and T remain deductively closed sets of sentences that are possibly not axiomatizable). Instead of Sent($\mathcal{L}$), Miller now takes the *models of* $\mathcal{L}$ as starting-point of his considerations. To be precise, Miller takes the quotient algebra Mod($\mathcal{L}$) modulo elementary equivalence, M^*, as his new point of departure. One element of M^* represents a set of elementary equivalent models. Two models are elementary equivalent if they verify (and falsify) the same sentences. Regarding first order logic M^* has the power of the continuum. A class $K \subseteq M^*$ of $\mathcal{L}$-models is an *elementary class* iff there is an $\mathcal{L}$-sentence σ such that $K = \mathrm{Mod}(\sigma)$. An intuitively plausible fact of mathematical logic reads that the elementary classes (*EC*) together form a Boolean algebra that is isomorphic to the Lindenbaum algebra of $\mathcal{L}$. Furthermore, as an *EC* is closed under finite intersections (and all its elements can be extended to filters), it forms a base for a topology on M^*. Under this topology, if $\mathrm{Mod}(A) \subseteq M^*$ is closed, it corresponds to a deductively closed set of sentences (an intersection of elementary classes), and if Mod(A) is open, it corresponds to the union of elementary classes. An element of M^* corresponds to a complete set of sentences, and only if Mod(A) is closed and open (clopen) it corresponds to a finitely axiomatizable system; that is a sentence (and Mod(A) is an elementary class).

$F \subseteq \mathcal{B}$ is a *filter* in $\mathcal{B}$ if for all x, y in F, $x \wedge y \in F$, and for all $x \in F$ and $y \in \mathcal{B}$, $x \vee y \in F$. An *ultrafilter* is a filter that is maximal with respect to the inclusion relation. An extension of an ultrafilter is equal to $\mathcal{B}$. The Stone space $S(\mathcal{B})$ of a Boolean algebra $\mathcal{B}$ is the power set algebra of the ultrafilters of $\mathcal{B}$, such that $\mathcal{B}$ is a subalgebra of $S(\mathcal{B})$. There is an isomorphism between EC and the Lindenbaum algebra $\mathcal{B}$ on the one hand, and EC is a base for the topology of M^* on the other. Consequently, M^* is homeomorphic with the Stone space of the Lindenbaum algebra $\mathcal{B}$ (or EC).[15] Miller chooses $\wp(S(EC))$ to represent the power set algebra of M^*, and calls this power set $\mathcal{S}$. He identifies a system or theory A with Mod(A), and measures the distances between the subsets of M^* by the downward strictly monotone autometric on $\mathcal{S}$. Miller already proved that if * is a normal downward strictly monotone autometric operation on a Boolean algebra, then $B * A = A \Delta B$. Thus, also in $\mathcal{S}$, Mod(A) * Mod(T) is equal to Mod(A) Δ Mod(T). Finally, note that the Mod-function does not embed the Brouwerian algebra $\mathcal{A}$ homeomorphically in $\mathcal{S}$. Although Mod($A \vee_{\mathcal{A}} B$) = Mod(A) $\cup_S$ Mod(B), and Mod($A \wedge_{\mathcal{A}} B$) = Mod(A) $\cap_S$ Mod(B), recall that Mod($\neg_{\mathcal{A}} T$) = Mod($\mathcal{L}$) which differs from Mod($\mathcal{L}$) $- \{\mathfrak{W}\} = -_S$Mod(T) where $\mathfrak{W}$ designates the complete and unaxiomatizable truth. Consequently, Mod($A \; \Delta_{\mathcal{A}} \; \neg_{\mathcal{A}} B$) $\neq$ Mod(A) Δ_S Mod(B). Miller's Stone space approach suggests the following verisimilitude definition ($M^* :=$ Mod($\mathcal{L}$)/$\equiv$).

DEFINITION 2.1: Let Mod(A), Mod(B) $\subseteq M^*$ and let the truth be represented by Mod(T) = $\{\mathfrak{W}\}$, $\mathfrak{W} \in M^*$. Then, the theory *B is closer to the truth T than A* if and only if: Mod(B) Δ $\{\mathfrak{W}\}$ < Mod(A) Δ $\{\mathfrak{W}\}$
Notation: $B <_T^{\Delta} A$

In other words, the theory B is closer to the truth than A if and only if the symmetric difference of B and the truth is a subset of the symmetric difference of A and the truth.

Although the Mod function embeds $\mathcal{A}$ somewhat heteromorphously in $\mathcal{S}$, the limiting results (4) and (5) remain in place. As mentioned before, according to Miller the second result is the most disturbing one. It shows that for axiomatizable theories, approaching the truth is the same as approaching every arbitrary theory X that is not axiomatizable. He tries to make the latter result more acceptable by comparing approaching the truth with searching for some point T situated among other points X, Y, Z etc. not far from the north pole on a globe. The north pole represents the tautology. If someone wants to reach T, and he is not too close to the north pole, then it does not matter whether he goes towards X, Y, Z or T. Heading north is heading in the right direction. According to Miller, the fact that the searching starts far more to the south than at the location where X, Y, Z or T are situated, reflects the fact that, generally, axiomatizable theories are much weaker than complete theories.[16] At the end of his paper, Miller claims that although his definition may be too weak (according to our convention in subsection 1.5.1), in his opinion, all other definitions are demonstrably too strong. We shall come to Miller's reservations in Chapter 5.

2.3. KUIPERS'S CONTENT DEFINITIONS

Theo Kuipers started to think about verisimilitude in structuralist terms in 1980, and he proposed definitions in terms of Suppes-Sneed structures. I base this presentation of Kuipers's ideas mainly on his 1982, 1987, and 1992 publications.[17] Kuipers distinguishes between content (naive) and refined definitions on the one hand, and comparative and quantitative ones on the other. The latter distinction is the usual one, and, regarding comparative definitions, the former contrast is by and large the same as the difference between content and likeness definitions. The present section deals with Kuipers's content definition. In subsection 2.3.1, I explain how Kuipers paraphrases a theory in structuralist terms. In subsections 2.3.2–2.3.3 we shall encounter Kuipers's comparative and quantitative proposals; Kuipers's refined definitions will be introduced in Chapter 3.

2.3.1. Kuipers's Theory Representation

In this subsection, we start with the introduction of Kuipers's way to represent scientific theories. Next, we illustrate Kuipers's terminology using his own light bulb example, and we end with some remarks and observations. Kuipers's intuitive and modal paraphrase of a theory is straightforward and clear. Let some conceptual framework $\mathcal{L}$ include all concepts relevant to some empirical problem. Then, a *theory is a claim according to which some conceptual possibilities are physically possible while the others are physically impossible.* Kuipers uses the Suppes-Sneed structuralism to formalize this basic point of departure concerning theories. In the structuralist framework, a theory representation is a two-tiered affair. It consists of a *set theoretical predicate*, on the one hand, and of the set of *intended applications* on the other. I call them the *conceptual part* and the *application* of a theory. The distinction is pivotal for Kuipers's verisimilitude definitions.

According to the structuralists, the first part is a universe of models or set theoretical structures M_p. It comprises all relevant conceptual possibilities. For our purposes, when comparing definitions using a Lindenbaum algebra, we may equate the structuralist M_p and Mod($\mathcal{L}$).[18] The second major part of the structuralist representation, is the set of intended applications I. The elements of I are (descriptions of) systems (that is, sets of objects or situations) in the real world that are the subject of the theory's claims. For example, the intended application of the simple balance mechanics is the set of all possible physical balances, or if the conceptual part consists of Newton mechanics, then I is the class of isolated sets of physical bodies. In another example, if M_p is the set of logical possible situations for a language about the weather conditions, I represents all places in the world that are subject to the meteorological theory. As for the distinction between theoretical and observational terms, I only consider the unstratified structuralist representation of scientific theories. The Ramsey technique used to formulate more sophisticated scientific claims, will be kept outside the discussion. It would only unnecessarily complicate our comparison.

Whereas I is a class of systems in the world, $M_p(I)$ is the *conceptual representation* of this class. It is the M_p-related true theory, or shortly, the truth. Obviously, not every set of conceptual possibilities combined with every class of intended applications will generate an *interesting* true theory $M_p(I)$. This is the reason for Kuipers to introduce the *frame-hypothesis* which claims that I in combination with M_p has to generate "a unique, time-independent subset $M_p(I) = T$ of all representations of the members of" I.[19] In other words, T is time independent. At least regarding the natural sciences this does not seem an unreasonable assumption. Technically speaking, this subset always exists, since the concepts of M_p apply to the intended applications, and therefore extensions of these functions and relations make up the true theory. Whether this true theory is an *interesting* one depends, among other things, on the question whether the extension of some functions and relations are non-empty. For example, as far as we know, the true combustion theory formulated in terms of phlogiston claims that the extension of the phlogiston concept is empty.

Kuipers differentiates between hypotheses and theories.[20] The *theory* X is paraphrased by a subset X of M_p accompanied with the claim that $X = T$, whereas $Y \subseteq M_p$ accompanied with the claim that $T \subseteq Y$ formalizes a *hypothesis* Y. Thus, the truth conditions for hypotheses are weaker than those for theories.[21] "Theory X is true or false when its claim '$T = A$' is true or false, respectively." ... "Hypothesis X is true or false when its claim '$T \subseteq X$' is true or false, respectively." Thus Y ($\subseteq M_p$) is *true as a hypothesis,* iff all $y \notin Y$: $y \notin T$ (y is empirically impossible), and $X(\subseteq M_p)$ is *true as a theory*, iff X is true as hypothesis and for all $x \in X$: $x \in T$ (x is physically possible). In other words, a *hypothesis* is true if all the conceptual possibilities it excludes are physically impossible. A theory is true not only if its excluded conceptual possibilities are indeed physically impossible, but also all the admitted conceptual possibilities are physically possible. Consequently, there is only one true theory in M_p, and there are many true hypotheses (laws) in M_p. Furthermore, a true theory is also a true hypothesis, and a false hypothesis is also a false theory. Occasionally, I will call hypothetical truth and a theoretical truth, *weak* and a *strong* truth, respectively.[22]

Kuipers not only distinguishes between strong and weak truth, he also separates *descriptive* from *theoretical* truth.[23] If a proposition is correct about *the actual* state of affairs, then it is descriptively true. If it is correct about *the set* of all physically possible states, it is theoretically true. These concepts are related to some language or conceptual framework $\mathcal{L}$ containing the concepts used to describe the relevant cognitive problem. Consequently, according to Kuipers, there is *descriptive*, and *theoretical* verisimilitude, and he will concentrate on theoretical verisimilitude as scientific theories try to capture all physical possibilities rather than only describe the actual state of affairs.

Kuipers also uses some model theoretical terminology. By definition, Y will be called a *consequence* of X iff all conceptual possibilities included by X are also included by Y, then, X is a subset of Y. We may also write $Y \in Q(X)$ in which $Q(X) := \{Y \mid X \subseteq Y \subseteq M_p\}$, the set of all consequences of X.[24] Reinterpreting Popper model theoretically, Kuipers identifies the empirical content of X by the set of conceptual possibilities that are excluded by X. This is of course X^c, the complement of X. Also in good Popperian spirit according to Kuipers X is *at least as falsifiable as* Y iff $|M_p - X| \geq |M_p - Y|$. This obtains when X is stronger than Y—that is $X \subseteq Y$.

According to Kuipers, $x \in X^c$ is a *potential counterexample* of the theory X.[25] He calls the elements of $X^c \cap T = T - X$ the *realizable* counterexamples, and those elements in $X \cap T^c = X - T$, the *virtual* counterexamples. The former are the physical possibilities wrongly excluded by X, and the latter the conceptual possibilities wrongly admitted by X. The set of all counterexamples, or falsifiers, of the claim $X = T$ is $(X - T) \cup (T - X)$ which will be abbreviated by $X \Delta T$. The set of realizable examples is of course the intersection of X and T, $X \cap T$. The set of all virtual examples, $X^c \cap T^c$, is the intersection of the complements of X and T. The next examples illustrate the terminology.

EXAMPLE: *(the electric light bulb)*. Kuipers often illustrates his terminology using an electric switching circuit. His light bulb example concerns a switching circuit connecting a battery and a light bulb.

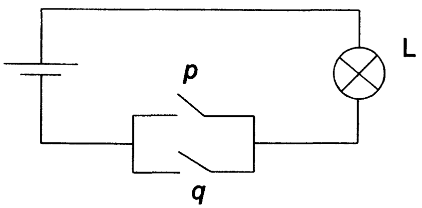


Fig. 2. A (simplified) circuit

Figure 2 represents a simple version of such a circuit example.[26] L is the light bulb, p and q are switches, and the object at the left is a battery. The switches are in a black box, and the light, battery, and connecting wires are visible. Let $\mathcal{L}[L, p, q]$ be a three propositional language interpreted by the light bulb circuit. Then M_p is equal to $\mathrm{Mod}(\mathcal{L}[L, p, q])$, and it is immediately clear that $\neg L \wedge \neg p \wedge \neg q$ is the descriptive truth of the example. We also see that $T = L \leftrightarrow (p \vee q)$ is the theoretically true theory, since $\text{Mod}(T) = \{\langle 1,1,1\rangle, \langle 1,1,0\rangle, \langle 1,0,1\rangle, \langle 0,0,0\rangle\}$ is the set of all physical possibilities (assuming that the battery is charged and the light bulb is intact). $L \rightarrow (p \vee q)$ is an example of a true hypothesis. It rightly excludes all conceptual possibilities of its complement. The following table displays Kuipers's use of all the other truth-values in this example.[27]

Table 2. Kuipers's light bulb example

Propositions	*weakly (almost usual)*	*strongly (new)*
Descriptively (complete)	*True*: $\text{Cn}(\neg L \wedge \neg p \wedge \neg q)$ *False*: $\text{An}(L \vee p \vee q)$	*True*: $\neg L \wedge \neg p \wedge \neg q$ *False*: $\text{Prop}(\mathcal{L}) - (\neg L \wedge \neg p \wedge \neg q)$
Theoretically (incomplete)	*True*: $\text{Cn}(L \leftrightarrow (p \vee q))$ *False*: $\text{Prop}(\mathcal{L}) - \text{Cn}(L \leftrightarrow (p \vee q))$	*True*: $L \leftrightarrow (p \vee q)$ *False*: $\text{Prop}(\mathcal{L}) - (L \leftrightarrow (p \vee q))$

As to the falsifiers, it is easy to confirm that the weak-theoretical falsehood $L \leftrightarrow p$ has two potential falsifiers. $\langle 1,0,1\rangle$ is the only realizable falsifier, and $\langle 0,0,1\rangle$ is the only virtual one. Contrary to the claim of $L \leftrightarrow p$, with p open and q closed, the light is on, and with those switch settings L cannot be off. Let us conclude the example by noting that the weak truth-values almost coincide with the usual valuations in logic. The only difference is that if the truth τ is incomplete, the false propositions are those in $\text{An}(\neg\tau)$ instead of $\text{Prop}(\mathcal{L}) - \text{Cn}(\tau)$. Of course the strong truth-values are all new. *End Example*

EXAMPLE: (*the coloured raven*). Let M_p consists of the predicates *Red, Black, Grey* and *Yellow*, and let I_X consist of all ravens. Moreover, let the true theory be that "all ravens are black or grey and there are black and grey ravens". Let the theory X claim that "all ravens are black or red, and there are ravens with such colour". Then, black ravens are realizable, and yellow ravens are virtual examples of the claim $X = T$. Grey ravens are realizable, and red ravens virtual counter examples of this claim. Finally, a yellow raven is a potential counter example of X, and a virtual example of the claim $X = T$. *End Example*

In the preceding, I have introduced Kuipers's paraphrase of theories and laws in structuralist terms, and illustrated his terminology using his light bulb example. In the next paragraphs I shall point to some consequences of Kuipers's theory representation. The first consequence of Kuipers's terminology is the possibility of a proposition φ, *and* his negation $\neg\varphi$ to be both false in the weak sense. In the example, the proposition $\varphi = L \wedge p \wedge q$ is false on the theoretical level, since it erroneously excludes $L \wedge p \wedge \neg q$; however, the negation of φ, $\neg L \vee \neg p \vee \neg q$, is also false in the weak sense, since it wrongly excludes $L \wedge p \wedge q$.

Thus, the preceding taxonomy of truth-values admits that φ *and* $\neg\varphi$ *are both false* in the (weak) *logical sense* (and therefore in the strong sense). That φ, and $\neg\varphi$ both can be (descriptively and theoretically) false in the strong sense is admissible, since it concerns a newly introduced concept, but that this can occur in the weak sense is, to say the least, remarkable.

The second remarkable implication of the preceding terminology concerns the *strength* of *theories*. It turns out that the *conceptual* strength is inversely proportional to the *empirical* strength of theories, and if both are equally important, the strengths of different theories cannot be compared. I have already mentioned that the structuralist theory representation is two-tiered. It consists of a theoretical predicate X, and a set of intentional applications I_X. *Conceptual strengthening* of hypothesis X consists in *restricting* the set of logical models of X; therefore for a hypothesis X' that is conceptually stronger than X, it follows that $X' \subseteq X$ (†). *Strengthening by application* of X consists in extending I_X. In terms of M_p-elements, the second method results in *extending* $M_p(I_X)$; therefore $M_p(I_X) \subseteq M_p(I_{X'})$ (‡). Since the empirical claim of hypotheses reads $M_p(I_X) \subseteq X$, (†) and (‡) may both apply.

A problem arises, however, if X and X' are *theories*, and their claims read: $X = M_p(I_X)$, and $X' = M_p(I_{X'})$. Consequently, regarding theories, if (†) and (‡) both apply, and the claims of X and X' are correct, then $X = X'$. Consequently, strictly speaking, according to the current theory representation, no theory is *conceptually* and *applicatively* stronger than X. Consider the light bulb example. There, $X' = L \rightarrow p$ implies $X = L \rightarrow (p \vee q)$, and X' is conceptually stronger than X. On the level of the applications, however, the opposite obtains. X is empirically stronger than X'; X takes more risks by allowing a larger set of physically possible situations. X (rightly) claims that p may be closed while L gives light—that is $\langle 1,0,1 \rangle$ is physically possible. This physical possibility is (wrongly) excluded by X'. Thus, the empirical claims of theories make their conceptual strength inversely proportional to their empirical strength.[28]

For the moment, I equate the strength of a theory with its logical strength, or the extent of the set of implied hypotheses. The set of hypotheses that are true according to the stronger theory includes the set of hypotheses that are true according to the weaker one. As to the applications of theories, I postpone the discussion to Chapter 4, which is dedicated to the problem of the epistemological question about approach to the truth.

A modal analysis of the foregoing shows that, even if M_p is equal to Mod($\mathscr{L}$) and X = Mod(χ), we need to distinguish between the model theoretical claim "Mod(χ) $\subseteq$ Mod($\mathscr{L}$) represents a theory" and the structuralist claim "$X \subseteq M_p$ paraphrases a theory". Let $\mathscr{L}$ be a finite propositional language, and let the constituents C_i represent its models. Then, the *model theoretical* claim that χ is a theory is tantamount to the claim that χ is a *disjunction* of constituents (the second part is redundant)

$$\chi := \vee_i \{C_i\} \ (\wedge [\wedge_j\{\neg C_j\}]).$$

Here, C_i describe the elements of Mod(χ), and C_j the elements of the complement of Mod(χ). The C_j need not be added as χ is false on all models of the constituents not implying χ; however, the *structuralist claim* that X is a theory implies that X is a *conjunction* of all physically possible constituents *and* physically impossible ones

$$X := \wedge_i\{\Diamond C_i\} \wedge \wedge_j\{\neg\Diamond C_j\}$$

Again the C_i describe the elements of X, and C_j describes those of X^c. Apart from the difference between being disjunctions and conjunctions, in contrast with the structuralist claim, the model theoretical one leaves no freedom of choice regarding the C_j. In a non-modal set-up the choice of C_i fixes the C_j.[29] In due course, we shall see that the structuralist approach about physical (in)possibilities requires a modal approach.

To sum up, within one specific context of relevant concepts M_p, the unstratified structuralist representation (p. 44) of the theory X consists of two parts:

1. The logical or *conceptual* part asserts that the conceptual possibilities excluded by X are physically impossible, and those that are included are physically possible. Strengthening the conceptual part of X means a *decrease* of X.
2. The *application* part, delineating the intended applications of the theory X, I_X.

Strengthening the intended applications of X implies an increase of $M_p(I_X)$, and via the empirical claim, $M_p(I_X) = X$, an *increase* of X. For the time being, I equate the strength of a theory with its conceptual or logical strength.

2.3.2. Structuralist Verisimilitude

Kuipers aims at a definition of theoretical approach to truth. Roughly speaking, Popper's intuitions about the theory Y being more verisimilar than X can be comprised as follows, Y includes the true claims of X, and X includes the false claims of Y. Kuipers's paraphrase of Popper's intuitions reads, all the admitted conceptual mistakes of Y are also conceptual mistakes of X, $Y-T \subseteq X-T$, and all the rightly included physical possibilities of X are also included by Y, $T-Y \subseteq T-X$. Or in Kuipers's terms: "Y has less (realizable and virtual) counterexamples than X has". Consequently, Kuipers gives the following comparative verisimilitude definition.[30]

DEFINITION 2.2: Let X, Y, $T \subseteq M_p$. Then, the theory Y is at least as close to the truth T as the theory X, $NTL(X,Y,T)$, if and only if: $Y \Delta T \subseteq X \Delta T$.
Notation: $Y \leq_T^{\Delta} X$

Or in terms of sets of models: $\psi \leq^{\Delta} \varphi :=_{def} \text{Mod}(\psi) \ \Delta\ \text{Mod}(\tau) \subseteq \text{Mod}(\varphi) \ \Delta\ \text{Mod}(\tau)$. Note that $Y \leq_T^{\Delta} X$ presupposes $I_X = I_Y = I$, and therefore that $M_p(I_X) = M_p(I_Y) = M_p(I) = T$. In other words, two theories can only be competitors in the struggle for being the closest to the truth if the frame-hypothesis applies and the sets of intended applications are identical.

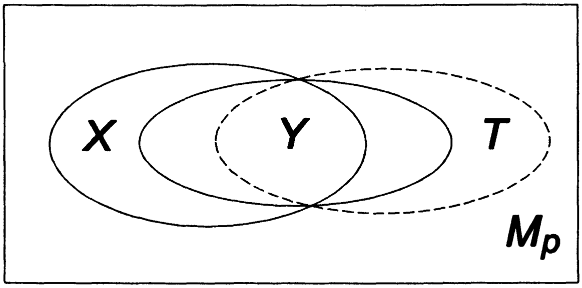


Fig. 3. Kuipers's Δ-definition

Kuipers bifurcates definition 2.2 into two clauses. According to his modal interpretation the first clause says that X excludes the physical possibilities that Y excludes $T-Y \subseteq T-X$ or $Y \cap T \supseteq X \cap T$. Kuipers calls this clause the *instantial* clause (Ni). It is equivalent to the constraint that $(X \cap T) - Y = \varnothing$. I call it the *antecedence clause*, the An-clause (Section 2.4). According to the modal interpretation, the second clause claims that all the conceptual failures of Y are also made by X; $Y-T \subseteq X-T$. Kuipers calls the latter the *explanatory* clause (Nii). He reformulates it by $Q(T) \cap Q(Y) \supseteq Q(T) \cap Q(X)$, and it is equal to the restriction $Y - (X \cup T) = \varnothing$. The name of this clause alludes to the fact that $Q(Y)$ is the set of all consequences of Y. All the true hypotheses explained by X, are also explained by Y. I call (Nii) the *consequence clause*, the Cn-clause, as it concerns the logical implications of the theories. A deeper understanding of the relation between the Cn-clause and the An-clause will come about after the algebraic analysis of the symmetric difference measure in Chapter 6.

Formally, Kuipers's definition is the same as Miller's definition 2.1 (page 42) which takes the Boolean algebra of the quotient space M^* as point of departure. Regarding the finite case, the only difference is that Miller assumes that the truth is *complete*, which is equal to the assumption that the truth is an atom in the algebra. The modal interpretation claims, however, that when the truth is represented by a singleton, we are dealing with the problem of approach to the *descriptive* truth whereas we ought to concentrate on theoretical truth. From the latter point of view, the truth being a singleton is just a borderline case. Nonetheless, $Y \leq_{\{t\}} X$ is well defined, and it suffers from the same drawbacks as Miller's definition. In the remaining text, if the completeness of the truth is irrelevant, I call it the Δ-definition, or the symmetric difference measure.

Finally, it may be worthwhile to remark that the Δ-definition treats all false but complete answers on the same footing. This is an important characteristic of content definitions. If the truth is complete (and finitely axiomatizable), then there is no pair of other complete theories whose verisimilitude can mutually be compared. This is due to the constraint that, regarding the similarity between formal propositions, content definitions only take the logical strength and the truth-value into account. Since the verisimilitude of constituents is uncomparable, Kuipers calls content definitions, naive definitions.

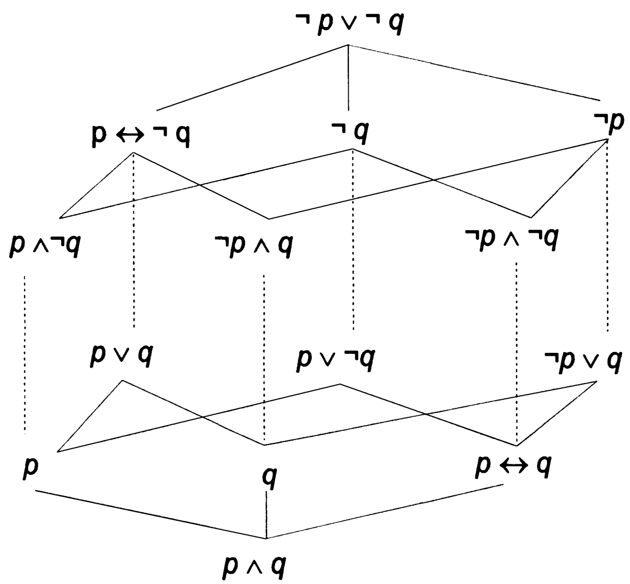


Fig. 4. $\leq^{\Delta}$-ordering of Prop($\mathscr{L}[p,q]$)

Figure 4 presents the Hasse-diagram of the $\leq^{\Delta}$-ordering of Prop($\mathscr{L}[p,q]$). The lowest element is the closest to the truth; if two elements are linked by a line, the subordinate is more verisimilar than the superior. The verisimilitude of elements that are not connected is uncomparable. The diagram shows that the verisimilitude of the false constituents is uncomparable. The Cn-clause establishes the ordering along the continuous lines; it orders the false propositions (top-layer) and the true ones (lower-layer); the An-clause fixes the ordering along the dotted lines. Diagrams such as in Figure 4, will help us to understand the important differences between the various definitions presented in this study.

2.3.3. *The Quantitative Content Definition*

Kuipers also gives a quantitative version of his verisimilitude definition. It is brought about by substituting the symmetric difference measure by comparing the cardinalities of the symmetric differences. Thus, the content distance *between* X and Y equals the size of the symmetric difference between both sets:

$$\mathrm{NTD}(X,T) := |X \Delta T| = |X-T| + |T-X|$$

The function NTD: $M_p \times M_p \to \mathbb{R}$ is a semimetric on M_p (p. 30). Kuipers designates $|X-T|$, the distance *from* X to T, by $\mathrm{NTD}(X\backslash T)$. Note that, as in the qualitative situation, the size of $X \cap T$ (relative to $X \cup T$) does not contribute to this content measure of distance. Kuipers's quantitative content definition now reads:[31]

DEFINITION 2.3: Let X, Y and T be subsets of M_p. The theory Y *is quantitatively at least as verisimilar as* X, iff $\mathrm{NTD}(Y,T) \leq \mathrm{NTD}(X,T)$
Notation: $Y \leq_T^{Dn} X$

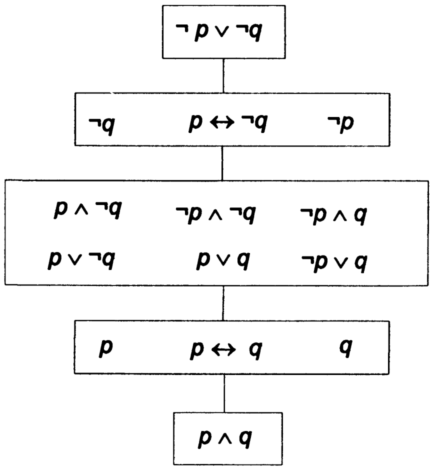


Fig. 5. The $\leq_T^{Dn}$- ordering of Prop($\mathscr{L}[p, q]$).

Trivially, for any pair X and Y, $Y \leq_T^{\Delta} X$ implies $Y \leq_T^{Dn} X$. The quantitative content definition is stronger than the qualitative one; it relates more theories. For instance, all false constituents receive the same distance to the truth. The relation between the two definitions is as follows. The qualitative definition claims that $(X \cap T)-Y$ and $Y-(X \cup T)$ are empty, whereas the quantitative definition only demands that

$$|(X \cap T)-Y| + |Y-(X \cup T)| \leq |(Y \cap T)-X| + |X-(Y \cup T)|$$

Figure 5 displays the quantitative content ordering of the Prop($\mathcal{L}[p, q]$). An element in the ordering is a set of propositions with the same distance to $p \wedge q$.

2.4. COMPARING THE CONTENT DEFINITIONS

Although Miller and Kuipers disagree about the completeness of the truth, their formal definitions are equivalent. I designate the symmetric difference ordering by $\leq^{\Delta}$. While presenting Miller's definition, I mentioned the fact if the truth is axiomatizable, then the symmetric difference measure is an appropriate measure on propositions (Lindenbaum algebra) and models (Stone space). The next proposition gives four different descriptions of the Δ-definition. Suppose $\mathcal{L}$ is a formal language with axiomatizable empirical truth τ.

PROPOSITION 2.1: If φ, ψ, and τ be elements of Prop($\mathcal{L}$), Then
$\mathrm{Mod}(\psi)\ \Delta\ \mathrm{Mod}(\tau) \subseteq \mathrm{Mod}(\varphi)\ \Delta\ \mathrm{Mod}(\tau) \Leftrightarrow$
a. $\mathrm{Cn}(\varphi) \cap \mathrm{Cn}(\tau) \subseteq \mathrm{Cn}(\psi)$ and b. $\mathrm{An}(\varphi) \cap \mathrm{An}(\tau) \subseteq \mathrm{An}(\psi)$

Proof: $\mathrm{Mod}(\psi)\ \Delta\ \mathrm{Mod}(\tau) \subseteq \mathrm{Mod}(\varphi)\ \Delta\ \mathrm{Mod}(\tau) \Leftrightarrow$ 1.a $\mathrm{Mod}(\psi)-\mathrm{Mod}(\tau) \subseteq \mathrm{Mod}(\varphi)$ and b. $\mathrm{Mod}(\tau)-\mathrm{Mod}(\psi) \subseteq \mathrm{Mod}(\tau)-\mathrm{Mod}(\varphi) \Leftrightarrow$ 2.a $\mathrm{Mod}(\psi) \subseteq \mathrm{Mod}(\varphi) \cup \mathrm{Mod}(\tau)$ and b. $\mathrm{Mod}(\varphi) \cap \mathrm{Mod}(\tau) \subseteq \mathrm{Mod}(\psi) \Leftrightarrow$ 3.a $\psi\ (\vee\ \tau) \vDash \varphi \vee \tau$ and b. $\varphi \wedge \tau \vDash \psi\ (\wedge\ \tau)$ (since $\mathrm{Cn}(\varphi \vee \psi) = \mathrm{Cn}(\varphi) \Leftrightarrow \cap\, \mathrm{Cn}(\psi)$, (†) and $\mathrm{An}(\varphi \wedge \psi) = \mathrm{An}(\varphi) \cap \mathrm{An}(\psi)$, (‡)) $\Leftrightarrow$ 4 a. $\mathrm{Cn}(\varphi) \cap \mathrm{Cn}(\tau) \subseteq \mathrm{Cn}(\psi)$ and b. $\mathrm{An}(\varphi) \cap \mathrm{An}(\tau) \subseteq \mathrm{An}(\psi)$. ⊠

The proof uses Kuipers's proposal to use the reflexive formulation and to distinguish between a. the explanatory clause and b. the instantial clause. Clearly, clause 3.a and (†) together imply that a. is equivalent to the Cn-clause $\mathrm{Cn}(\varphi) \cap \mathrm{Cn}(\tau) \subseteq \mathrm{Cn}(\psi)$. Consequently, the Δ-definition is strictly weaker than the consequence definition: for all propositions ψ and φ: if $\psi \leq^{\Delta} \varphi$, then $\psi \leq^{+} \varphi$.[32] On the other hand, if $\tau \equiv p \wedge q$, $\psi \equiv p \wedge \neg q$, and $\varphi \equiv p \vee q$, then $\psi \leq^{+} \varphi$ and $\psi \nleq^{\Delta} \varphi$. Further, 3.b and (‡) together imply that $\mathrm{An}(\varphi) \cap \mathrm{An}(\tau) \subseteq \mathrm{An}(\psi)$, and that is the reason for calling b. the antecedence clause. In brief, the Δ-definition replaced the falsity clause by the antecedence clause. Chapter 6 contains a more extensive study into the relation between the $\leq^{\Delta}$- and the $\leq^{+}$-ordering. The next observation gives the shortest syntactical expression of the Δ-definition.

Observation 2.2: For ψ, φ and $\tau \in$ Prop($\mathcal{L}$), ψ is at least as close to τ as φ, that is $\psi \leq^{\Delta}_{\tau} \varphi$, iff $\varphi \leftrightarrow \tau \vDash \psi \leftrightarrow \tau$
Proof: $\psi \leq^{\Delta}_{\tau} \varphi \Leftrightarrow$ [(†) $\psi \vee \tau \vDash \varphi \vee \tau$ and (‡) $\varphi \wedge \tau \vDash \psi \wedge \tau$] $\Leftrightarrow$
[(†) $\tau \vee (\neg\varphi \wedge \neg\tau) \vDash \tau \vee (\neg\psi \wedge \neg\tau)$ and (‡) $\neg\tau \vee (\varphi \wedge \tau) \vDash \neg\tau \vee (\psi \wedge \tau)$] $\Leftrightarrow$
[(†) $\neg\varphi \vee \tau \vDash \neg\psi \vee \tau$ and (‡) $\varphi \vee \neg\tau \vDash \psi \vee \neg\tau$] $\Leftrightarrow$
[(†) $\varphi \rightarrow \tau \vDash \psi \rightarrow \tau$, and (‡) $\tau \rightarrow \varphi \vDash \tau \rightarrow \psi$] ⊠

Until now, I have not presupposed the completeness of the empirical truth. The next observation, however, only obtains under this assumption.

PROPOSITION 2.3: Let τ be the complete truth of Prop($\mathscr{L}$). Then for all $\psi, \varphi \in$ Prop($\mathscr{L}$): $\psi \leq_{\tau}^{\Delta} \varphi \Leftrightarrow [((\tau \rightarrow \psi) \wedge (\psi \rightarrow \varphi)) \vee ((\psi \rightarrow \varphi) \wedge (\varphi \rightarrow \neg\tau)) \vee \{(\tau \rightarrow \psi) \wedge (\varphi \rightarrow \neg\tau) \wedge (\psi \vDash (\varphi \vee \tau))\}]$

Proof: Let $\psi \leq_{\tau}^{\Delta} \varphi$; since τ is complete, we need only consider four truth-value distributions for ψ and φ. 1. Let $\varphi, \psi \in Cn(\tau)$; then 3.a, b of prop. 2.1 show that $\psi \leq_{\tau}^{\Delta} \varphi$ reduces to $\psi \vDash \varphi$. 2. Let $\varphi, \psi \in An(\neg\tau)$; then 1.a, b of prop. 2.1 show that $\psi \leq_{\tau}^{\Delta} \varphi$ also reduces to $\psi \vDash \varphi$. 3. Let $\psi \in Cn(\tau)$, $\varphi \in An(\neg\tau)$; then 3.a, b of prop. 2.1 show that $\psi \leq_{\tau}^{\Delta} \varphi$ reduces to $\psi \vDash \varphi \vee \tau$. 4. Let 4. $\varphi \in Cn(\tau)$, $\psi \in An(\neg\tau)$; then 2.b of prop. 2.1 implies $\psi \not\leq_{\tau}^{\Delta} \varphi$. This proves the proposition, as these are all possible situations. ⊠

The preceding proposition gives a good intuition of the Δ-definition when the truth is complete. It shows that under that circumstance the Δ-definition equals the implication relation for propositions with the same truth-value; if ψ and φ have different truth-values, then $\psi \leq_{\tau}^{\Delta} \varphi$ equals $\psi \vDash \varphi \vee \tau$. Thus, the verisimilitude of two propositions of the same logical strength and truth-value, is uncomparable. Another property of the Δ-definition is the preference of the contradiction to all consistent false propositions.

Observation 2.4: 1. For all consistent false $\varphi \in$ Prop($\mathscr{L}$): $\perp <_{\tau}^{\Delta} \varphi$; 2. for all consistent true $\varphi \in$ Prop($\mathscr{L}$): $\varphi <_{\tau}^{\Delta} \top$.

Proof: 1. If $\varphi \in An(\neg\tau)$, then $Mod(\tau)\ \Delta\ Mod(\perp) = Mod(\tau) \subset Mod(\varphi)\ \Delta\ Mod(\tau)$. 2. If $\varphi \in Cn(\tau)$, then $Mod(\varphi)\ \Delta\ Mod(\tau) = Mod(\varphi) - Mod(\tau) \subset Mod(\top)$; and $\varphi <_{\tau}^{\Delta} \top$. ⊠

Discovering whether a theory is a contradiction (or a tautology) is an a priori affair; as the contradiction and the tautology have no empirical content, we drop them as serious approach-to-the-truth candidates. Next we prove that using the Δ-content definition we cannot compare any φ with its negation if $\varphi \not\equiv \tau$.

Observation 2.5: $\psi \leq_{\tau}^{\Delta} \neg\psi$ iff $\psi \equiv \tau$

Proof: $\Leftarrow$: Trivially true. $\Rightarrow$: Let $\psi \leq_{\tau}^{\Delta} \neg\psi$; then $\psi \vDash \neg\psi \vee \tau$ which implies $\psi \vDash \tau$. In addition $\neg\psi \wedge \tau \vDash \psi$ which implies $\tau \vDash \psi$ therefore $\psi \equiv \tau$. ⊠

The preceding proposition claims that according to the Δ-definition the verisimilitude of an empirical theory cannot be compared with that of its negation; except if it is the truth. This is a remarkable feature of the Δ-definition. For example, if $p \wedge q$ is the truth, $p \wedge q \wedge r \not\leq_{\tau}^{\Delta} \neg p \vee \neg q \vee \neg r$. It is the An-clause that is responsible for this truth-value feature since $p \wedge q \wedge r \leq_{\tau}^{+} \neg p \vee \neg q \vee \neg r$; however the difference only matters if the compared theories have different truth-values. In practice, where all theories are likely to be false, the difference is not relevant. Furthermore, as Miller's objection 5 (p. 41) shows, if the truth is complete but *not axiomatizable*, then the An-clause evaporates and only a larger truth content suffices to be closer to the truth. Under these special circumstances, the Δ-definition is equivalent to the consequence definition, and they are both equivalent to the logical deduction relation.

At the end of subsection 2.2.2, I referred to the next remarkable consequence of the Δ-definition in case the truth is not axiomatizable. Recall that Cnst(φ) is the set of all constituents that imply φ.

PROPOSITION 2.6: If the truth is complete and not axiomatizable, then no true proposition is Δ-comparable with any false proposition.

Proof: Assume that proposition ψ is true, proposition φ is false and $\psi \leq^{\Delta} \varphi$. Then there is a true $\beta \in \mathrm{Cnst}(\psi)$ with $\beta \notin \mathrm{Cnst}(\varphi)$. Since the truth is complete and not axiomatizable there is a $p \notin \mathrm{voc}(\psi) \cup \mathrm{voc}(\varphi)$ such that p or its negation is true. Assume the truth of p. Certainly, $\beta \wedge \neg p \vDash \psi$ and $\beta \wedge \neg p \nvDash \varphi$ and since $\beta \wedge \neg p$ is false and not contradictory there is a model $\mathfrak{M} \in \mathrm{Mod}(\beta \wedge \neg p)$ such that $\mathfrak{M} \in \{\mathrm{Mod}(\psi)\ \Delta\ \{\mathfrak{W}\}\}$ and $\mathfrak{M} \notin \{\mathrm{Mod}(\varphi)\ \Delta\ \{\mathfrak{W}\}\}$ (if $\neg p$ were true substitute p for $\neg p$). Hence, $\psi \nleq^{\Delta} \varphi$, and a contradiction arises; therefore not $\psi \leq^{\Delta} \varphi$. ⊠

2.5. THE PROPERTIES OF THE CONTENT DEFINITIONS

Metatheoretical properties of approach-to-the-truth definitions were introduced in Chapter 1 and these properties will be instrumental for the assessment of the three content definitions, viz. the consequence definition, and Miller's and Kuipers's versions of the Δ-definition. In this section, we assume that the complete truth of language $\mathscr{L}$ is axiomatizable.[33] Before discussing the metatheoretical properties of the first chapter, we first mention the nature of the $\leq^{\Delta}$-ordering. In subsection 1.4.1 we saw that $\leq^{+}$ preorders Prop($\mathscr{L}$) (observ. 1.4, p. 20). Here we show that Miller's and Kuipers's addition of the An-clause makes $\leq^{\Delta}$ a partial ordering.

Observation 2.7: $\leq^{\Delta}$ is a (weak) partial ordering of Prop($\mathscr{L}$).
Proof: The reflexivity and transitivity of $\leq^{\Delta}$ immediately follows from Observation 2.2 (p. 52). Now for the antisymmetry, assume that $\psi \leq^{\Delta}_{\tau} \varphi$ and $\varphi \leq^{\Delta}_{\tau} \psi$; then, the same observation implies $\varphi \leftrightarrow \tau \equiv \psi \leftrightarrow \tau$; this implies $\varphi \rightarrow \psi$ and $\psi \rightarrow \varphi$, and therefore $\psi \equiv \varphi$. ⊠

PROPOSITION 2.8: For all $\tau \in \mathrm{Prop}(\mathscr{L})$ $\leq^{\Delta}_{\tau}$ is a lattice on Prop($\mathscr{L}$).

Proof: (together with Krabbe): Let for arbitrary ψ, φ and τ, $Y := \mathrm{Mod}(\psi)$, $X := \mathrm{Mod}(\varphi)$ and $T := \mathrm{Mod}(\tau)$. Since we know that $\leq^{\Delta}$ is a (weak) partial ordering on

Prop($\mathscr{L}$) we only need to show that X and Y have 1. a supremum, and 2. an infimum.

1. Let us define the supremum of X and Y by $X \vee_\Delta Y := ((X \cup Y) - T) \cup (X \cap Y \cap T)$ (the union of X and Y 'outside T' and intersection 'inside T'). We have to verify whether a. $X \leq_\tau^\Delta X \vee_\Delta Y$, b. $Y \leq_\tau^\Delta X \vee_\Delta Y$, and c. For all Z with $X \leq_\tau^\Delta Z$ and $Y \leq_\tau^\Delta Z$ holds $X \vee_\Delta Y \leq_\tau^\Delta Z$. First notice that $(X \vee_\Delta Y) \Delta T = (T-(X \cap Y)) \cup ((X \cup Y) - T)$. *a.* Since $(T-X) \cup (X-T) \subseteq (T-(X \cap Y)) \cup ((X \cup Y)-T)$, $X \leq_\tau^\Delta X \vee_\Delta Y$ obtains. *b.* For reasons of symmetry $Y \leq_\tau^\Delta X \vee_\Delta Y$ also holds. *c.* Let $X \leq_\tau^\Delta Z$ and $Y \leq_\tau^\Delta Z$ obtain for some Z; then $(X \Delta T) \cup (Y \Delta T) \subseteq (Z \Delta T)$, and therefore $(T-(X \cap Y)) \cup ((X \cup Y)-T) \subseteq (Z \Delta T)$; thus $(X \vee_\Delta Y) \Delta T \subseteq (Z \Delta T)$. Let us turn to the argument concerning the infimum.

2. Let us define the infimum of X and Y by $X \wedge_\Delta Y := ((X \cup Y) \cap T) \cup ((X \cap Y) - T)$ (the intersection of X and Y 'outside T' and union 'inside T'). Then $(X \wedge_\Delta Y) \Delta T = (T-(X \cup Y)) \cup ((X \cap Y)-T)$; therefore *a.* $X \wedge_\Delta Y \leq_\tau^\Delta X$ since $(T-(X \cup Y)) \cup ((X \cap Y)-T) \subseteq (T-X) \cup (X-T)$; *b.* for reasons of symmetry $X \wedge_\Delta Y \leq_\tau^\Delta Y$ holds. Finally, if for some Z, $Z \leq_\tau^\Delta X$ and $Z \leq_\tau^\Delta Y$ hold, then $Z \Delta T \subseteq (X \Delta T) \cap (Y \Delta T) = (X \wedge_\Delta Y) \Delta T$, which comes down to $Z \leq_\tau^\Delta X \wedge_\Delta Y$. Thus $\leq_\tau^\Delta$ is a lattice on Prop($\mathscr{L}$). ⊠

2.5.1. The Δ-definition and Popper's revision are content definitions

Irrespective of the completeness of the truth, according to the Δ-definition and the +-definition, there is no theory worse than $\neg\tau$ as for all $\varphi \in$ Prop($\mathscr{L}$) applies: $\neg\tau \leftrightarrow \tau \vDash \varphi \leftrightarrow \tau$. In Miller's model, in which all propositions are positioned on a globe, τ and $\neg\tau$ are *antipodes*. All other propositions scattered on the globe are between truth and falsehood. Consequently, the Δ- and +-definition are content definitions.

Proposition 2.3 (p. 53) shows why the two definitions are also subject to the child's-play objection. The first two disjuncts show that for two theories with the same truth-value, the Δ-definition equals the +-definition. Thus, the two content definitions claim that, if ψ implies a false φ, then it is closer to the truth than φ. Consequently, if the theory φ is false, then all theories $\varphi \wedge \chi$ are closer to the truth than φ; even if χ is a fairy tale.

2.5.2. Truth-value dependency

According to the Δ-definition, no false theory is better than any true theory (see proposition 2.3). In terms of Section 1.5 this means that if the truth τ is axiomatizable, then the Δ-definition *is truth-value dependent*, and the consequence definition is truth-value *in*dependent. The latter is obvious since, if $\varphi \in$ Cn(τ), $\psi \in$ An($\neg\tau$), and $\psi \vDash \varphi$, then, according to the consequence definition, ψ is at least as close to the truth as φ. As mentioned before (p. 41), this difference between Popper's revision and the Δ-definition disappears, if the truth is not axiomatizable, since then the truth-content of φ fails to be axiomatizable, and the two definitions are identical.

2.5.3. Specularity

At the end of Chapter 1, I defined specularity. A verisimilitude ordering $<_\tau$ has the *specularity property* if for all ψ and φ of Prop($\mathscr{L}$), $\psi <_\tau \varphi \Leftrightarrow \neg\psi <_{\neg\tau} \neg\varphi$. Strictly speaking only Kuipers's content definition is specular, and Miller's is not, since Miller assumes that the truth is complete. If we skip this condition we may conclude that the Δ-definition is specular. Let the truth of $\mathscr{L}$ be axiomatizable; then, for all ψ and φ of Prop($\mathscr{L}$) $\psi \leq^\Delta_\tau \varphi \Leftrightarrow \neg\psi \leq^\Delta_{\neg\tau} \neg\varphi$ since, obviously, $\varphi \leftrightarrow \tau \vDash \psi \leftrightarrow \tau$ is necessary and sufficient for $\neg\varphi \leftrightarrow \neg\tau \vDash \neg\psi \leftrightarrow \neg\tau$. In these elementary circumstances, besides Millers assumption about the completeness of the truth, the consequence definition lacks the specularity property for another reason. $\psi \vDash \varphi \vee \tau$ (†) is not equivalent to $\neg\psi \vDash \neg\varphi \vee \neg\tau$ (‡). For example, consider the case in which φ and τ are true and ψ is false. Then, (†) is true, and (‡) is false. It is the An-clause that makes the Δ-definition specular.

2.5.4. Context dependence

In this subsection, we consider the context dependence of the $\leq^\Delta$-ordering, and in Chapter 6 (subsection 6.3.4) we will focus on the context dependency of the +-ordering.

PROPOSITION 2.9: The $\leq^\Delta$-ordering is *weakly context independent.*

Proof: Let $\varphi, \psi \in$ EmProp($\mathscr{L}$) and $\varphi \not\equiv \psi$ for some $\mathscr{L}$ with complete truth τ. Let $\mathscr{L}' \supseteq \mathscr{L}$ be some expansion of $\mathscr{L}$ with complete truth τ' and let $\psi <^\Delta_\tau \varphi$. We have to prove that $(\varphi \leftrightarrow \tau) \vDash (\psi \leftrightarrow \tau)$ implies $(\psi \leftrightarrow \tau') \nvDash (\varphi \leftrightarrow \tau')$ ($\vDash$ is the deduction relation in $\mathscr{L}$ and in $\mathscr{L}'$). Evidently, if ψ is true and φ is false, the truth-value dependency of $<^\Delta_\tau$ excludes $\varphi <^\Delta_{\tau'} \psi$. Consequently, we need only consider the case in which $\psi <^\Delta_\tau \varphi$ equals $\psi \vDash \varphi$, ψ and φ having the same truth-value (see observ. 2.2, p. 52). Since $\varphi \not\equiv \psi$ and $\psi \vDash \varphi$, $\exists\mathfrak{M} \in$ Mod($\mathscr{L}'$) such that $\mathfrak{M} \vDash \varphi$, $\mathfrak{M} \vDash \neg\psi$ and since $\psi <^\Delta_\tau \varphi$, $\mathfrak{M} \nvDash \tau$. As $\mathfrak{M} \nvDash \tau'$ ($\tau' \vDash \tau$), and $\mathfrak{M} \nvDash \psi$, $\mathfrak{M} \vDash \psi \leftrightarrow \tau'$ and $\mathfrak{M} \nvDash \varphi \leftrightarrow \tau'$; therefore $(\psi \leftrightarrow \tau') \nvDash (\varphi \leftrightarrow \tau')$. ⊠

EXAMPLE: In $\mathscr{L}[p]$, p is closer to p than $\neg p$: $p \leq^\Delta_p \neg p$. After all $(\neg p \leftrightarrow p) \vDash (p \leftrightarrow p)$. In the expansion $\mathscr{L}'[p,q]$, p and $\neg p$ are uncomparable (≭) according to the Δ-definition:

$$(p \wedge q) \vee (p \wedge \neg q) \quad \not\asymp^\Delta_{p\wedge q} \quad (\neg p \wedge q) \vee (\neg p \wedge \neg q)$$

since $\langle 1,0\rangle \in \mathrm{Mod}(p)\ \Delta\ \mathrm{Mod}(p \wedge q)$ and $\langle 0,1\rangle \notin \mathrm{Mod}(\neg p)\ \Delta\ \mathrm{Mod}(p \wedge q)$. The same holds for the logical expansion of $p \leq_p \neg p$ into $\mathscr{L}_{S5}[p]$ which in modal constituents reads:

$$(p \wedge \Diamond\neg p) \vee (p \wedge \neg\Diamond\neg p) \not\lessgtr^{\Delta}_{p \wedge \Diamond\neg p} (\neg p \wedge \Diamond p) \vee (\neg p \wedge \neg\Diamond p)$$

End Example

Although the Δ-definition is weakly context independent, the preceding example shows that the Δ-definition is not *strongly* context independent; and the example is no coincidence. Generally, if $\psi \leq^{\Delta}_{\tau} \varphi$, and ψ is true and φ false, then an expansion of the language, and the completeness of the truth in the new language, together create a possibility for a new, false, constituent to imply ψ but not to imply φ. For instance, although $p <_{p\wedge q} \neg q$ in $\mathscr{L}[p, q]$, p and $\neg q$ are uncomparable regarding the truth $p \wedge q \wedge r$, $p \not\lessgtr_{p \wedge q \wedge r} \neg q$, in $\mathscr{L}[p, q, r]$ since $p \wedge q \wedge \neg r$ implies both p and $\neg q$ but is not a constituent of the truth. The same holds for modal expansions. An expansion of the logical vocabulary, however, will not reverse the original $\leq^{\Delta}$-ordering.

2.5.5. *Disappearance of the falsifity clause*

In the Δ-definition, the Cn-clause is more important than the An-clause. First, proposition 2.3 (p. 53) shows that, if τ is complete, and ψ and φ have the same truth-value, then the Δ-definition is equivalent to Popper's revision; and if the truth is not axiomatizable, even these truth-value conditions can be cancelled. Differences between the Cn- and the Δ-definition come about if the truth τ is incomplete; e.g. if $\varphi \vDash \psi \vDash \tau$. Then, the An-clause manages to render φ less verisimilar than ψ whereas according to the +-definition the verisimilitude of φ and ψ is the same; φ and ψ have the same set of true consequences. Further, the ordering impact of the An-clause is on the truth-value of the theory, not on the truth-values of its consequences. It prohibits a false theory to be closer to the truth than a true one, even if the truth is complete, and φ is logically much stronger than ψ. In short, the An-clause is not the full-fledged falsity clause that Popper wanted his verisimilitude definition to possess. For all these reasons, it is more appropriate to consider the Δ-definition as a sophisticated version of The consequence definition than as a mature successor of Popper's original proposal. In a way, accepting the Δ-definition is tantamount to dropping Popper's original demand that the better theory can exceed the merits of its predecessor, and avoid some of its drawbacks, if the latter has been paraphrased by false consequences. This conclusion fits well to my non-monotonic diagnosis in the first chapter. The combination of the Cn-clause with a full-fledged falsity clause requires a non-monotonic system.

2.5.6. Complete Truth Discussion

In the approach-to-the-truth project, there is an apparent controversy about the complete truth assumption. The two representatives of the Δ-definition disagree about the question whether the truth is complete. Kuipers explicitly claims that "empirical theories are not, and should not be, complete."[34] Niiniluoto maintains that "this formulation is misleading."[35] Miller assumes that the truth is complete. To keep the account as simple as possible, again, I shall confine the discussion of the differences of opinion to the finite propositional case. In the sequel, completeness will be used in the following sense. The theory τ is complete with regards to some language $\mathcal{L}$ iff:

$$\forall \varphi \in \mathrm{Prop}(\mathcal{L})\ \tau \vDash \varphi \ \text{ or } \ \tau \vDash \neg\varphi$$

or, which comes down to the same, all the models of τ are elementary equivalent. From Miller and Popper's point of view, approach to the truth has to be based on logical strength and truth-value. Thus, the verisimilitude of a proposition can only be fixed if it has a definite truth-value. Kuipers's proposal showed that Miller's definition can also be applied if the truth is not complete; however, if propositions without a definite truth-value are ordered using the symmetric difference definition, verisimilitude resembles many-valued logic. The paradigmatic examples of Kuipers's content definition are those in which $\mathrm{Mod}(X) \cap \mathrm{Mod}(T) \neq \varnothing$ and $\mathrm{Mod}(X) \cap \mathrm{Mod}(T)^c \neq \varnothing$. Those theories X lack a definite truth-value. Consequently, one can defend the claim that Kuipers's content definition is a variant of (comparative) many-valued logic. It establishes a partial ordering of truth-values. Popper and most other participants in the project want to avoid this "many-valued logic" character. Thus, when defining verisimilitude, they assume a complete truth, and use a classical and definite language.

Kuipers's argument in favour of an incomplete truth starts from another point of view.[36] His basic consideration is that a scientific law has something to do with all *physically possible* states of affairs. He claims that the complete truth is identical to a complete description of the actual state of affairs. Consequently, approach-to-the-truth definitions assuming a complete truth explicate approach to the *descriptive* truth. Kuipers considers approach to the *theoretical* truth to be more relevant for science than descriptive truthlikeness, since science pursues knowledge about all physically possible states of affairs instead of only the actual one.[37] According to the structuralists, the truth is typically a set of structures, and the truth as a singleton is exceptional. Kuipers sometimes adds the following line of argument to illustrate his point of view. Scientific laws must be *weak* propositions, since they do not fix a particular state of affairs about some realistic system without adding initial conditions. For instance, Newton's laws do not forecast the exact orbit of Neptune without adding initial information about its mass, distance to the sun, and so forth. It pinpoints descriptive information about particular systems only in combination with these initial conditions.

Kuipers's argument in favour of incompleteness of the truth contains a fallacy. His argument has the following scheme:

1. the (theoretical) truth is physical realizability of $\mathscr{L}$-structures
2. more than one $\mathscr{L}$-structures can be realized
3. a complete theory is true only on one $\mathscr{L}$-structure
——————————————— [therefore]
4. the truth is not complete

In this paraphrase of the argument the notion "truth" is clearly ambiguous; therefore 4. does not follow from 1., 2. and 3. In reality, the truth of 1. is modal whereas the truth of 4. is not; in one modal model a number of possible structures may be accessible (compare the situation of the light bulb example). Consequently, we must distinguish between the truth of $\mathscr{L}$, and the truth of some related modal language, for instance, $\mathscr{L}_{S5}$. The truth about the realizability (the $\mathscr{L}_{S5}$-truth) *is* complete in the sense that it fixes for *all* structures of $\mathscr{L}$ whether it is realizable or not. The truth of $\mathscr{L}_{S5}$ *is* therefore complete in the language $\mathscr{L}_{S5}$. The real scheme of Kuipers's argument, therefore reads:

1*. the truth of $\mathscr{L}_{S5}$ is physical realizability of $\mathscr{L}$-structures
2. more than one $\mathscr{L}$-structure can be realized,
3*. a complete $\mathscr{L}_{S5}$-theory is true only on one $\mathscr{L}_{S5}$-model
——————————————— [therefore]
4*. the $\mathscr{L}_{S5}$-truth is not complete

There is no incompatibility between the statement that the truth of $\mathscr{L}_{S5}$ is complete and that more than one $\mathscr{L}$-structure is realizable according to that complete S5-truth. Consequently, 4* does not follow from 1*, 2 and 3*. In the next section I shall formulate Kuipers's modal verisimilitude intuitions in a S5-language. Then, we shall see that the new modal formulation of the Δ-definition manages to combine Miller and Kuipers's point of view about the truth.

Finally, Kuipers (1992) does not mention the incompleteness of the truth. In private conversation, however, Kuipers maintains that in his new approach all theories are "total" in the following sense. The theories assert about all conceptual possible worlds, whether it is physically possible or impossible. As we have seen in subsection 2.3.1 this means that no theory can be *conceptually* and *applicatively* stronger than another theory. Whatever the consequences of this last point of view might be, if M_p is a universe of possible worlds, Kuipers maintains that the truth must be represented by a subset of those possible worlds, whereas Miller claims that the truth is supposed to be a singleton.

2.6. TWO MODAL PROPOSALS

Now that the important content definitions have been presented, in this last section I propose two modal content definitions, which retain the merits of the existing content proposals and avoid some of their drawbacks. In the first subsection, I summarize some problems of Δ-definition. Next, I shall introduce some formal preliminaries. In subsection 2.6.4, I formulate Kuipers's ideas about an adequate content definition in modal terms. This results in the *double symmetric difference* definition, which solves the problems of Kuipers's formal content proposal, and does not suffer from the child's-play argument. In the fourth subsection, I show that we can add a full-blooded falsity clause to a modal reformulation of the +-definition. This final modal proposal improves the double symmetric difference definition.

2.6.1. Difficulties

As we saw in Section 2.5, the Δ-definition has to cope with serious problems. The following three are the most important ones. First, regarding the Δ-definition, it is a *child's play* to improve a falsified theory. Every antecedent of a false theory φ is closer to the truth than φ itself. Second, it is incomprehensible that scientists propose highly abstract theories since the latter are very likely to be false, and, due to the An-clause of the Δ-definition, false theories are never closer to the truth than a true theory; the Δ-proposal is *truth-value dependent*. Third, if the compared propositions have the same truth-value, and the truth is complete, which is the normal situation, the *Δ-definition reduces to the Cn-proposal*, and therefore fails to explain Popper's emphasis on falsification. From a likeness point of view, we could even add that the content definition does not consider the similarity between possible worlds. Miller, however, does not consider this a disadvantage since it leads to the "language dependent" approach to the truth.

As Kuipers's and Miller's content proposals are formally the same, they suffer from the same drawbacks. Kuipers, however, sketches an interesting, intuitive modal interpretation of the symmetric difference definition. Inspired by the structuralist approach, he recognizes the importance of the distinction between the conceptual part, and the empirical part of a theory. Although this intuitive description is an important contribution, its formal presentation has two undesirable implications. First, Kuipers paraphrases his modal intuitions using an *incomplete truth* (subsection 2.5.6). Consequently, his proposal displays features of many-valued logic. Kuipers uses at least two different notions of truth and falsity, and according to these truth-values, a proposition *and its negation* can be false (in the normal and strong sense). Second, the modal interpretation *prohibits the comparison of the strength* of empirical theories. As shown in subsection 2.3.1, if the theory X is logically stronger than the theory Y, $X \subseteq Y$, then X cannot be empirically stronger than Y, $Y \subseteq X$. Thus, the formal definition fails to do justice to the intuitive appealing separation of the conceptual and empirical parts of a theory. In the ideal situation, the conceptual and empirical parts are independent.

We are not alone in criticizing Kuipers's use of structures. Niiniluoto also objects to Kuipers's use of possible worlds.[38] He mentions two possible solutions to the problem. The first is to change Kuipers's paraphrase of a theory into

$$(\pm)\Diamond\alpha_1 \wedge \ldots \wedge (\pm)\Diamond\alpha_i \wedge \ldots \wedge (\pm)\Diamond\alpha_n$$

in which α_i are constituents of some non-modal formal language $\mathscr{L}$. As it emphasizes the physical possibility of the structures, this proposal is inspired by the instantial clause. Niiniluoto's second proposal is to replace the non-modal constituents in the original definition by nomic constituents,

(6) $\bigwedge_i\Diamond\alpha_i \wedge \alpha_j \wedge \Box(\bigvee_i\alpha_i)$.[39]

This is the way Niiniluoto incorporates the criticism of Cohen according to which science strives at (true) *laws* instead of sheer truth.[40]

My modal proposals start with Niiniluoto's second suggestion. This enables us to combine ideas about physical (im)possibilities and the structuralist two-tiered representation of theories. Additionally, we can restore both clauses of Popper's original definition. Preserving the advantages of the preceding content definitions, the modal approach avoids most of the known content drawbacks. Moreover, it even explains some shortcomings of the Δ-definition. First, let us consider the modal preliminaries, which form the basis for both modal definitions.

2.6.2. The Modal Preliminaries

The coming modal approach-to-the-truth definitions order the propositions of a finite modal propositional language $\mathscr{L}_{S5}[p_1, ..., p_n] \supseteq \mathscr{L}[p_1, ..., p_n]$, or $\mathscr{L}_{S5}$ in short. $\mathscr{L}_{S5}$ is the conservative modal extension of $\mathscr{L}$. Prop($\mathscr{L}_{S5}$) designates the set of all $\mathscr{L}_{S5}$ sentences modulo logical equivalence. Semantically, the modal models of $\mathscr{L}_{S5}$, Mod($\mathscr{L}_{S5}$) are defined by $\mathfrak{M} := \langle W, R, V\rangle$ in which

W is a set of possible worlds
$R \in W \times W$; (in S_5, R is an equivalence relation)
V is an unambiguous valuation assignment for all $p_i \in$ voc($\mathscr{L}$) on all $w \in W$.[41]

In the context of finite S5 languages, V defines a bijection between the elements of Cnst($\mathscr{L}$) and W of any $\mathfrak{M}$. The possible worlds in W must be distinguished from the set of the *modal* $\mathscr{L}_{S5}$*-models*. These models are described by the modal constituents $\gamma \in \mathrm{Cnst}^{\Box}(\mathscr{L}_{S5})$. The distinction between the set of possible worlds W and the set of modal models Mod($\mathscr{L}_{S5}$), plays an important role in the coming modal definitions. Since in S_5-languages R is an equivalence relation, it bifurcates W in a subset of accessible, and a subset of inaccessible worlds. For instance, in $\mathscr{L}_{S5}[p,q]$, $W = \{\langle 1,1\rangle, \langle 10\rangle, \langle 0,1\rangle, \langle 0,0\rangle\}$, and there are $4^2 - 1$ possible non-empty R-relations, and $\mathscr{L}_{S5}$-constituents. First, let us consider the modal constituents of $\mathscr{L}_{S5}$. An $\mathscr{L}_{S5}$-constituent is an enumeration of the actual world, and all possible and impossible worlds. In principle, it has the following form:

$$\gamma := \Diamond\alpha_1 \wedge \ldots \wedge \Diamond\alpha_{m-1} \wedge \alpha_m \wedge \neg\Diamond\alpha_{m+1} \wedge \ldots \wedge \neg\Diamond\alpha_n$$

in which $\{\alpha_1 \ldots \alpha_n\} = \mathrm{Cnst}(\mathscr{L})$ represents all possible worlds, the constituents of the corresponding language $\mathscr{L}[p_1,\ldots,p_n]$. The $\mathscr{L}_{S5}$-constituent consists of three parts. The first part, $\Diamond\alpha_1 \wedge \ldots \wedge \Diamond\alpha_{m-1}$, fixes the possible $\mathscr{L}$-constituents of the model. Second, α_m describes the actual world. Finally, $\neg\Diamond\alpha_{m+1} \wedge \ldots \wedge \neg\Diamond\alpha_n$ describes the words that are impossible. In other words, γ claims for all possible worlds whether it is accessible. As $\neg\Diamond p$ equals $\Box\neg p$, $\neg\Diamond\alpha_{m+1} \wedge \ldots \wedge \neg\Diamond\alpha_n$ equals $\Box\neg\alpha_{m+1} \wedge \ldots \wedge \Box\neg\alpha_n$, which equals $\Box(\neg\alpha_{m+1} \wedge \ldots \wedge \neg\alpha_n)$. Hence an $\mathscr{L}_{S5}$-constituent γ may be paraphrased by

(7) $\quad \gamma := \Box\lambda \wedge \Diamond\alpha_1 \wedge \ldots \wedge \Diamond\alpha_{m-1} \wedge \alpha_m \quad$ with $\lambda := \neg\alpha_{m+1} \wedge \ldots \wedge \neg\alpha_n$

since γ bifurcates Cnst($\mathscr{L}$), $\alpha_1 \vee \ldots \vee \alpha_m \equiv \lambda$. For incomplete ψ, however, $\alpha_1 \vee \ldots \vee \alpha_m$ only implies λ. For example, $\Box(p \vee q) \wedge (p \wedge q) \wedge \Diamond(\neg p \wedge q) \wedge \Diamond(p \wedge \neg q)$ is a constituent in which $\lambda \equiv p \vee q$; and $\psi := \Box(p \vee q) \wedge \Diamond(\neg p \wedge q) \wedge \Diamond(p \wedge \neg q)$ is an incomplete proposition where $(\neg p \wedge q) \vee (p \wedge \neg q) \vDash p \vee q$. It claims that $p \wedge \neg q$ and $\neg p \wedge q$, the "included worlds," are physically possible, and $\neg p \wedge \neg q$, the "excluded world," is physically impossible. $\mathrm{Wrd}^i(\psi)$ refers to the set of worlds excluded by ψ; and $\mathrm{Wrd}^e(\psi)$ designates the group of ψ-included worlds.

Let me be somewhat more precise. Let voc($\mathscr{L}$) = voc($\mathscr{L}_{S5}$), and let W be the domain of $\mathfrak{M} \in \mathrm{Mod}(\mathscr{L}_{S5})$, and finally let $\psi \in \mathrm{Prop}(\mathscr{L}_{S5})$. Then, the worlds included by ψ, are defined by: $\mathrm{Wrd}^i(\psi) := \{w \in W \mid w \vDash \alpha$ with $\alpha \in \mathrm{Cnst}(\mathscr{L})$, and $\psi \vdash \Diamond\alpha\}$; the worlds excluded by ψ, are defined by: $\mathrm{Wrd}^e(\psi) := \{w \in W \mid w \vDash \alpha$ with $\alpha \in \mathrm{Cnst}(\mathscr{L})$, and $\psi \vdash \neg\Diamond\alpha\}$.

Just as for the elements of Prop($\mathscr{L}$), all propositions of $\mathscr{L}_{S5}$ are equivalent to disjunctions of (modal) constituents γ_i. Thus if $\psi \in \mathrm{Prop}(\mathscr{L}_{S5})$, then

$$\vDash \psi \equiv \vee_i \gamma_i.$$

In principle, all S5-propositions may be interpreted to represent theories. Scientific theories, however, usually neglect the question which of the physically possible worlds is the actual one. Consequently, those elements of Prop($\mathscr{L}_{S5}[p,q]$) that only convey information about which $\mathscr{L}$-constituents are physically possible, and which are not, represents empirical theories; ψ of the preceding paragraph is a typical example of such a theory. As information about the actual world is considered irrelevant, in what follows we are only interested in comparing *scientific propositions* represented by elements of SProp($\mathscr{L}_{S5}$). The latter is defined by an equivalence relation on the modal constituents of $\mathscr{L}_{S5}$, Cnst$^{\Box}$($\mathscr{L}_{S5}$). Let us say that for all γ_i and γ_j of Cnst$^{\Box}$($\mathscr{L}_{S5}$), $\gamma_i \approx \gamma_j$ iff $\mathrm{Wld}^i(\gamma_i) = \mathrm{Wld}^i(\gamma_j)$ and $\mathrm{Wld}^e(\gamma_i) = \mathrm{Wld}^e(\gamma_j)$. Clearly, $\approx$ partitions Cnst$^{\Box}$($\mathscr{L}_{S5}$), and $\zeta := \vee_i \gamma_i$ for all $\gamma_i \in [\gamma_j]^{\approx}$ is an element of Cnst$^{\Box}$($\mathscr{L}_{S5}$)/$\approx$. This brings us to the definition of SProp($\mathscr{L}_{S5}$).

DEFINITION 2.4: Let $\approx$ partition the set of all modal constituents of $\mathscr{L}_{S5}$.Then, $\varphi \in$ SProp($\mathscr{L}_{S5}$) iff $\varphi \equiv \vee_j \zeta_j$ and all $\zeta_j \in$ Cnst$^{\Box}$($\mathscr{L}_{S5}$)/$\approx$

Intuitively, SProp($\mathscr{L}_{S5}$) equals Prop($\mathscr{L}_{S5}$) modulo "information about the actual world". In the same way as γ_i is complete in Prop($\mathscr{L}_{S5}$), ($\forall\varphi \in$ Prop($\mathscr{L}_{S5}$): $\gamma_i \vDash \varphi$ or $\varphi \vDash \neg\gamma_i$), $\zeta \equiv \vee_i \gamma_i$, (with $\gamma_i \in [\gamma_j]^{\approx}$) is complete in SProp($\mathscr{L}_{S5}$). I shall call ζ_j *total* in Prop($\mathscr{L}_{S5}$).

Now that I explained the representation of a scientific theory, we come to the representation of the truth. In the modal definitions that are to come, in principal, the empirical truth τ is the strongest Tarskian true proposition in $\mathscr{L}_{S5}$. Additionally, I assume that the language used is determinate, which means that τ is complete and all modal propositions receive an empirical truth-value. In other words τ is a modal constituent, $\tau \in$ Cnst$^{\Box}$($\mathscr{L}_{S5}$). Scientists, however, are often interested in the truth "modulo the actual situation", $[\tau]^{\approx}$; therefore, in what follows, I shall consider τ to be total rather than complete in Prop($\mathscr{L}_{S5}$). The truth τ of Prop($\mathscr{L}_{S5}$), then, claims for all worlds (or $\mathscr{L}$-constituents) w, whether it is physically possible or not. For example, in $\mathscr{L}_{S5}[p,q]$, τ can be equal to $\Box(p \vee q) \wedge \Diamond(p \wedge q) \wedge \Diamond(\neg p \wedge q) \wedge \Diamond(p \wedge \neg q)$.

Semantically, we may designate the complete empirical truth of Prop($\mathscr{L}_{S5}$) by one modal model $\mathfrak{M}$, modulo elementary equivalence, where Mod(τ) is equal to $\{\mathfrak{M}\}$. The complete truth τ of SProp($\mathscr{L}_{S5}$) may be represented by one modal model $\mathfrak{B}$, modulo elementary equivalence *and* modulo actual world information. $\mathfrak{B}$ bifurcates the set of conceptual possible worlds $w \in W$ into a subset of physically possible worlds, $\mathrm{Wld}^i(\tau) = \{w \in W |\ aRw\}$, a being the actual world, and a subset of physically impossible worlds $\mathrm{Wld}^e(\tau) = \{w \in W |\ w$ falsifies $\tau\}$; if τ is total in SProp($\mathscr{L}_{S5}$) the union of these two subsets equals W.

In brief, modal propositions represent empirical theories, and a modal constituent represents the empirical truth. Further, in what follows I neglect the question of which of the possible worlds is the actual one.

2.6.3. Reformulation of Kuipers's Intuitions; Double Symmetric Difference

The basic intuition of Kuipers's content proposal is that the true theory indicates for all logically possible worlds whether it is physically possible or impossible. Arbitrary theories also make claims about the physical (im)possibility of logical possible situations. The extent of the set of mistakes regarding excluding and including possible worlds establishes the verisimilitude of the theory. Consequently, regarding our modal reformulation of Kuipers's proposal only $(\pm)\Diamond\alpha_i$ represent crucial empirical data.

2.6.3.1 The 2Δ-definition

Following Kuipers's intuitions, I distinguish two kinds of corroboration or confirmation of the theory ψ. It consists of the physical possibilities rightly predicted by the theory (II), and the impossibilities rightly excluded by the theory (IV) (the Roman capitals refer to areas in Figure 6).

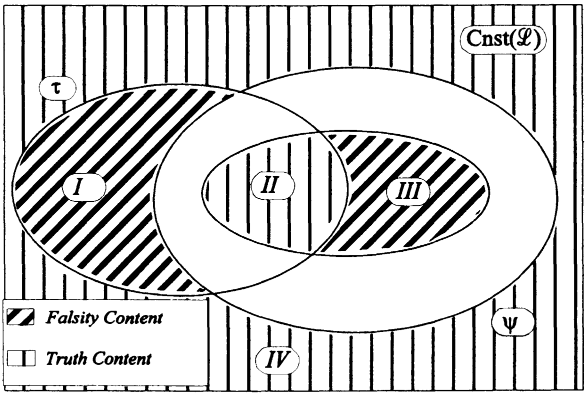


Fig. 6. Modal theory presentation

Thus, I define the *truth-content of the theory* ψ in terms of $\mathcal{L}$-constituents that ψ rightly includes (II) and those it rightly excludes (IV):

$$Ct_T^{\Diamond}(\psi) :=_{def} \{\alpha \in \mathrm{Cnst}(\mathcal{L}) | \ \tau \vDash \Diamond\alpha \text{ and } \psi \vDash \Diamond\alpha, \text{ or } \tau \vDash \neg\Diamond\alpha \text{ and } \psi \vDash \neg\Diamond\alpha\}$$

On the contrary, falsification of ψ consists in finding situations in which some true $\mathcal{L}$-constituent α ($\tau \vDash \alpha$) falsifies ψ ($\alpha \in$ I). Moreover, ψ's included impossibilities are also part of ψ's falsity content ($\alpha \in$ III). I define the *falsity-content of the theory* ψ in terms of $\mathcal{L}$-constituents that ψ wrongly excludes (I) and those it wrongly includes (III):

$Ct_F^{\Diamond}(\psi) :=_{def} \{\ \alpha \in \mathrm{Cnst}(\mathscr{L}) \mid \tau \vDash \Diamond\alpha$ and $\psi \vDash \neg\Diamond\alpha$, or $\tau \vDash \neg\Diamond\alpha$ and $\psi \vDash \Diamond\alpha\}$

The truth-content and the falsity content of the theory ψ is illustrated in Figure 6. Comparing Figure 3 (p. 49) with Figure 6 we clearly see the similarities and differences between Kuipers's classical proposal and our modal approach. An important difference concerns the logical strength of a theory. Here, the S5-logical strength of ψ is not the extent of $\mathrm{Wrd}^i(\psi)$, which is the set of $\mathscr{L}$-constituents included by ψ; nor is it $\mathrm{Wrd}^e(\psi)$, the excluded $\mathscr{L}$-constituents. Here, the logical strength is inversely proportional to the extent of the set of $\mathscr{L}$-constituents the theory is indifferent about—that is $\mathrm{Cnst}(\mathscr{L})- \{\mathrm{Wrd}^e(\psi)\cup\mathrm{Wrd}^i(\psi)\}$. In Figure 6 this is the area without hatching. Another difference concerns the truth-value of the modal propositions. Here, ψ is true iff $\mathrm{Wrd}^i(\psi) \subseteq \mathrm{Wrd}^i(\tau)$ and $\mathrm{Wrd}^e(\psi) \subseteq \mathrm{Wrd}^e(\tau)$; ψ is false if it is not true. Deliberately leaving out the intricacies regarding the truth of the actual world, I come to my modal reformulation of Kuipers's content definition:

DEFINITION 2.5: Let φ, $\psi \in \mathrm{Prop}(\mathscr{L}_{S5})$. Then, ψ is at least as verisimilar as φ iff

1. $Ct_T^{\Diamond}(\psi) \supseteq Ct_T^{\Diamond}(\varphi)$
2. $Ct_F^{\Diamond} F(\varphi) \supseteq Ct_F^{\Diamond}(\psi)$

Notation: $\psi \leq_{\tau}^{2\Delta} \varphi$

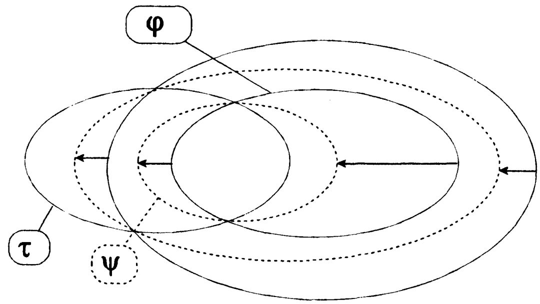


Fig. 7. $\psi \leq_{\tau}^{2\Delta} \varphi$ for false φ and ψ

Clause 1. of the definition claims $\mathrm{II}_\psi\cup \mathrm{IV}_\psi \supseteq \mathrm{II}_\varphi\cup\mathrm{IV}_\varphi$; and clause 2, claims $\mathrm{I}_\psi\cup\mathrm{III}_\psi \subseteq \mathrm{I}_\varphi\cup\mathrm{III}_\varphi$. Since for φ and ψ, $\mathrm{II} \cap \mathrm{IV} = \varnothing$ and $\mathrm{I} \cap \mathrm{III} = \varnothing$ (*), $\mathrm{II}_\psi \supseteq \mathrm{II}_\varphi$ and $\mathrm{III}_\varphi \supseteq \mathrm{III}_\psi$, which is equal to $\mathrm{Wrd}^i(\tau)\ \Delta\ \mathrm{Wrd}^i(\varphi) \subseteq \mathrm{Wrd}^i(\tau)\ \Delta\ \mathrm{Wrd}^i(\psi)$, (†). Moreover, (*) implies $\mathrm{IV}_\psi \supseteq \mathrm{IV}_\varphi$ and $\mathrm{I}_\varphi \supseteq \mathrm{I}_\psi$, therefore $\mathrm{Wrd}^e(\tau)\ \Delta\ \mathrm{Wrd}^e(\psi) \subseteq \mathrm{Wrd}^e(\tau)\ \Delta\ \mathrm{Wrd}^e(\varphi)$, (‡). I abbreviate the conjunction of (†) and (‡) by:

$$\mathrm{Wrd}^{i\&e}(\tau)\ \Delta\ \mathrm{Wrd}^{i\&e}(\psi) \subseteq \mathrm{Wrd}^{i\&e}(\tau)\ \Delta\ \mathrm{Wrd}^{i\&e}(\psi)$$

The double symmetric difference inclusion for two false propositions φ and ψ is illustrated in Figure 7.

Let us summarize the proposal in this subsection. In our modal reformulation the elements of Cnst($\mathscr{L}$) represent situations in the world that confirm or falsify a theory. I use the S5-modal extension of $\mathscr{L}$ to paraphrase theories. Neglecting the truth about the actual world, a complete theory makes claims about all elements of Cnst($\mathscr{L}$) whether it is physically possible or not. The theory ψ is closer to the truth than φ if it better agrees with τ about the (im)possibility of $\mathscr{L}$-constituents than φ does.

2.6.3.2 The Light Bulb Example under the 2Δ-definition

Let me illustrate my modal representation of Kuipers's content approach using his paradigmatic light bulb example.

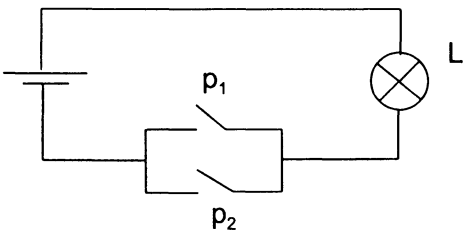


Fig. 8. A (simplified) circuit

Figure 8 represents the diagram of parallel switches, connecting a light bulb with a battery. Kuipers analyses the example using a classical propositional language. In his analysis the theoretical truth is $\tau = L \leftrightarrow (p \vee q)$ and the truth is not complete.[42] The descriptive truth is $\neg L \wedge \neg p \wedge \neg q$. Our modal reformulation also distinguishes these two concepts. The complete truth τ of Figure 8 is:

(8) $\tau := \Box(L \leftrightarrow (p \vee q)) \wedge$ (a)
$\neg L \wedge \neg p \wedge \neg q \wedge$ (b)
$\Diamond(L \wedge p \wedge q) \wedge \Diamond(L \wedge \neg p \wedge q) \wedge \Diamond(L \wedge p \wedge \neg q)$ (c)

in which (b) is the descriptive truth. Kuipers rightly claims that the descriptive truth is not the major object of scientific investigations. Theories are concerned with physical (im)possibilities. The theoretical truth about the diagram is

(9) $\tau := \Box(L \leftrightarrow p \vee q) \wedge$ (a)
$\Diamond(\neg L \wedge \neg p \wedge \neg q) \wedge \Diamond(L \wedge p \wedge q) \wedge \Diamond(L \wedge \neg p \wedge q) \wedge \Diamond(L \wedge p \wedge \neg q)$ (c)

which is an element of Prop($\mathscr{L}_{S5}$). The theoretical truth omits the information about the actual world (8.b). In our presentation of the modal definition, the question, which of the physically possible worlds is the actual one, does not affect the ordering of the SProp($\mathscr{L}_{S5}$) elements, which represent the scientific theories.

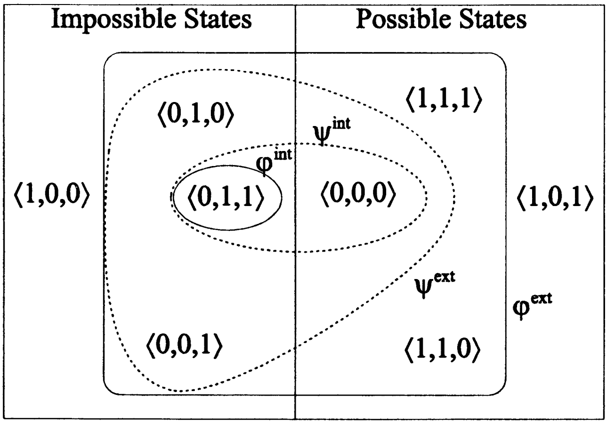


Fig. 9. No child's-play for 2Δ-definition

We show that, regarding to the child's play objection, and in contrast to Kuipers's content definition, not all antecedences of a false theory φ are more verisimilar than φ itself. Let ψ and φ be defined as follows (see Figure 9)

$$\psi := \Box(\neg L) \wedge \Diamond(\neg L \wedge \neg p \wedge \neg q) \wedge \Diamond(\neg L \wedge p \wedge q)$$
$$\varphi := \Box(\neg L \vee p) \wedge \Diamond(\neg L \wedge p \wedge q)$$

As φ and ψ rightly exclude $L \wedge \neg p \wedge \neg q$ as physically possible and ψ also rightly includes $\neg L \wedge \neg p \wedge \neg q$, the truth-content of ψ comprises the truth-content of φ:

$$Ct_T^{\Diamond}(\varphi) = \{L \wedge \neg p \wedge \neg q\} \subset \{\neg L \wedge \neg p \wedge \neg q, L \wedge \neg p \wedge \neg q\} = Ct_T(\psi)$$

Both propositions φ and ψ are false, $\psi \models \varphi$, and they wrongly exclude $L \wedge \neg p \wedge q$ and include $\neg L \wedge p \wedge q$. Regarding the falsity content, ψ is worse of than φ:

$$Ct_F^{\Diamond}(\varphi) = \{L \wedge \neg p \wedge q, \neg L \wedge p \wedge q,\} \subset \{\neg L \wedge p \wedge q, \; L \wedge p \wedge q, L \wedge \neg p \wedge q, \; L \wedge p \wedge \neg q\} = Ct_F^{\Diamond}(\psi)$$

The conclusion reads that theories φ and ψ are uncomparable according to the modal verisimilitude definition and not all false antecedences of φ are more verisimilar than φ. The theory φ has a smaller truth content and a smaller falsity content than the theory ψ. The child's-play argument that bothered the previous content definitions, does not impair our modal proposal. Some antecedences of the false ψ, however, are more verisimilar than ψ. Take for example $\psi' := \Box(\neg L \vee p) \wedge \Diamond(\neg L \wedge \neg p \wedge \neg q) \wedge \Diamond(\neg L \wedge p \wedge q)$. Moreover, this modal proposal does not fall victim to the Miller-Tichý argument.

2.6.3.3 The Consequences of the Modal Approach

In this subsection I summarize the improvements of the modal formalization of Kuipers's comparative content proposal, which are clustered in three groups. First, we shall look at some down-to-earth ordering properties of $\leq_\tau^{2\Delta}$. Then, we show that the double symmetric difference solves problems of the structuralist approach. Finally, we shall see that the modal approach solves some difficulties of the Δ-definition.

The $\leq_\tau^{2\Delta}$-ordering is a preorder of Prop($\mathscr{L}_{S5}$). Its reflexivity and transitivity follow directly from the definition. Since the $\leq_\tau^{2\Delta}$-ordering discards information about the actual world, it cannot order two propositions that only disagree about the actual world; the ordering is not antisymmetric on Prop($\mathscr{L}_{S5}$). The minima and maxima of the truth- and falsity-contents are $\varnothing$ and Cnst($\mathscr{L}$), respectively. Truth- and falsity-content of the tautology and contradiction in $\mathscr{L}_{S5}$ are $Ct_T^\Diamond(\top) = Ct_F^\Diamond(\top) = \varnothing$, $Ct_T^\Diamond(\bot) = \mathrm{Cnst}(\mathscr{L}) \cap \mathrm{Cnst}(\tau)$, and $Ct_F^\Diamond(\bot) = \mathrm{Cnst}(\mathscr{L}) - \mathrm{Cnst}(\tau)$. The modal proposal inherits the relatively good performance of the contradiction from the Δ-definition, according to which the contradiction is the best false proposition. This is not a serious drawback as we already removed the contradiction and tautology from the list of propositions bearing similarity to the empirical truth. In addition to the relatively high verisimilitude of the contradiction, the modal Δ-definition is truth-value dependent. As false propositions have a non-empty falsity-content, according to the present modal proposal, it cannot be more verisimilar than any true proposition with an empty falsity-content. Finally, we observe that according to the $\leq_\tau^{2\Delta}$-proposal the modal constituent ξ with $\mathrm{Wld}^i(\tau) = \mathrm{Wld}^e(\xi)$ and $\mathrm{Wld}^e(\tau) = \mathrm{Wld}^i(\xi)$ is the worst element and not $\neg\tau$. This likeness behaviour is due to the restriction of the consequences to those of the form $(\pm)\Diamond\alpha_i$ ($\alpha \in \mathrm{Cnst}(\mathscr{L})$). We encountered the same phenomenon in subsection 1.4.1 where we restricted the set of all consequences of a theory to the set of the implied literals.

My modal paraphrase solves at least two major problems of the structuralist approach. In the first place, it ends the discussion about the incompleteness of the truth as described in subsection 2.5.6. According to my modal reformulation the empirical truth is complete in the sense that it indicates for every $\mathscr{L}$-constituent whether it is physically possible or not; however, following Kuipers's intuitive descriptions, the present modal proposal marginalises the actual truth and focuses on the physical (im)possibilities. In the second place, the double symmetric definition gets around and even explains the difficulties concerning the strength of theories in the structuralist approach as explained in subsection 2.3.1. Increase and decrease of sets of models cannot both represent increasing logical strength. Our modal presentation shows that increase of sets of possible $\mathscr{L}$-constituents implied by φ, $\Diamond\alpha_i$, and increase of sets of φ-implied impossible $\mathscr{L}$-constituents, $\neg\Diamond\alpha_i$, both contribute to the increase of φ's logical strength.

The 2Δ-definition solves, in addition to the specific problems for the structuralist approach, the general problems of the Δ-definition. Comparing the Δ- and +-definition in Section 2.4, we saw that the Cn-clause overruled the falsification clause. Popper dropped the second altogether; and the An-clause of the Δ-definition is unable to explain the relevance of falsification. Our modal proposal restores the power of the falsity clause of Popper's original proposal. Without falling victim to Miller's and Tichý's argument, the truth-content and falsity clauses both independently exert their influences. In other words $Ct_T^{\Diamond}(\psi) \supset Ct_T^{\Diamond}(\varphi)$ does not exclude $Ct_F^{\Diamond}(\psi) \subset Ct_F^{\Diamond}(\varphi)$.

For example, let τ be defined by $\tau := \Box(p) \wedge \Diamond(p \wedge q) \wedge \Diamond(p \wedge \neg q)$ and let $\psi := \Box(\neg q) \wedge \Diamond(p \wedge \neg q)$ and $\varphi := \Box(\neg p \vee \neg q) \wedge \Diamond(\neg p \wedge \neg q) \wedge \Diamond(p \wedge \neg q) \wedge (\neg p \wedge q)\}$. Then $Ct_T^{\Diamond}(\psi) = \{p \wedge \neg q, \neg p \wedge q\} \supset \{p \wedge \neg q\} = Ct_T^{\Diamond}(\varphi)$ and $Ct_F^{\Diamond}(\psi) = \{p \wedge q\} \subset \{p \wedge q, \neg p \wedge \neg q, \neg p \wedge q\} = Ct_F^{\Diamond}(\varphi)$. Last but not least, the 2Δ-proposal blocks Tichý's child's-play argument. Take, for instance, τ and φ of the preceding example. Then, the antecedence φ of the false $\varphi' := \Box(\neg p \vee \neg q) \wedge \Diamond(p \wedge \neg q)$ is less verisimilar than φ'.

Summarizing the preceding observations, we may say that the modal paraphrase of the structuralist approach solves the problems of the latter concerning logical strength and incompleteness of the truth. Counterbalancing the importance of the mutually independent falsity- and truth-content of a theory, the modal paraphrase survives Miller and Tichý's argument that brought down Popper's original proposal, and blocks Tichý's child's-play argument. These findings confirm the contention that, contrary to a modal language, a classical language does not provide enough structure to paraphrase Kuipers's modal verisimilitude intuitions. Finally, honesty compels us to admit that the double symmetric difference has two drawbacks. First, the definition remains truth-value dependent; no false theory improves a true one, even if the latter is almost the tautology. Second, according to our first modal proposal the negation of the truth is not the worst proposition. In the next subsection, I propose an improved modal version of the Δ-definition.

2.6.4. The modal version of the consequence Definition

According to the formal paraphrase of Kuipers's content proposal of the preceding subsection, the verisimilitude of a modal proposition is based on the (in)correct inclusion and exclusion of possible worlds. This 2Δ-definition inherits the truth-value dependency of the standard Δ-definition. To retain the advantages of the modal approach, and avoid truth-value dependency, I choose a slightly different foundation for my final modal proposal. Intuitively, the *modal +-definition* reformulates the consequence definition in modal terms, creating the opportunity to add a full-blooded falsity clause based on non-modal possible worlds. Thus, instead of restricting ourselves only the non-modal constituents, here *all* modal consequences contribute to the truth-content of a theory, whereas the falsity content only pertains to non-modal constituents. In doing so, the proposal incorporates the difference of abstraction level between the accepted laws and theories at the one hand, and falsifying instantiations at the other.

Let us turn to the definition. First, just as in the preceding subsection, here, I paraphrase theories by modal propositions. Second, as the truth τ has to imply all relevant Tarskian-true propositions of a determinate language $\mathcal{L}_{S5}$, and it is not falsified by any physically possible situation, τ is a (modal) constituent.[43] Then, we may define the truth- and falsity-content of a modal proposition ψ as follows. The *truth-content of the theory* ψ is the set of all true modal consequences of ψ:

$$Ct_T^+(\psi) :=_{def} \{ \varphi \in \mathrm{Prop}(\mathcal{L}_{S5}) | \ \psi \vDash \varphi \text{ and } \tau \vDash \varphi\}$$

Clearly, the definition of the truth-content of ψ is the +-definition for modal languages. As the falsifying instances of accepted laws and theories have a different level of abstraction than the laws or theories, they are represented by possible worlds.

According the intuitive description about the truth, ψ is falsified, if we encounter an (accessible) world w of the ("true") world structure $\mathfrak{W}$ that falsifies ψ. Note that this falsification takes into account only *w and not the entire structure of* $\mathfrak{W}$. Thus, ψ is falsifiable iff $\exists w \in \mathfrak{W}$: $w \in \mathrm{Wrd}^e(\psi)$. In words, ψ is falsifiable if there is a physically possible world that falsifies ψ. Consequently, the *falsity-content of the theory* ψ is the set of accessible, or physically possible worlds that falsify ψ:

$$Ct_F^{\square}(\psi) :=_{def} \{w \in \mathfrak{W} \ | \ w \in \mathrm{Wrd}^e(\psi)\}$$

In one word, $\mathcal{L}$-constituents represent world-situations that may or may not occur, and which may confirm or falsify a theory (or law), which is represented by an $\mathcal{L}_{S5}$ proposition ψ. These truth- and falsity-contents suggest the next verisimilitude definition:

DEFINITION 2.6: Let $\psi, \varphi \in \mathrm{Prop}(\mathcal{L}_{S5})$. Then, according to the *modal consequence definition*, ψ is at least as verisimilar as φ iff
1. $Ct_T^+(\psi) \supseteq Ct_T^+(\varphi)$
2. $Ct_F^{\square}(\varphi) \supseteq Ct_F^{\square}(\psi)$

Notation: $\psi \leq_\tau^{\boxplus} \varphi$

Thus ψ is at least as close to the truth τ as φ iff (1.) all true consequences (or laws or predictions) of φ are also true consequences of ψ, and (2.) all physical possible situations that falsify ψ, also falsify φ. Again, I call 1. the Cn-clause—it is the +-definition in modal terms—and clause 2. the falsity clause. I have already claimed that the two clauses act on different levels. The Cn-clause concerns inclusions of *modal-models*, whereas the falsity clause pertains to *non-modal possible worlds*, which are elements of the modal-models. Let us consider some consequences of the modal consequence or ⊞-definition.

To start with, we examine some properties of the $\leq^{\boxplus}_{\tau}$-orderings relation. First, $\leq^{\boxplus}_{\tau}$ is a transitive and reflexive on Prop($\mathscr{L}_{S5}$); it is a preordering; however, $\leq^{\boxplus}_{\tau}$ is not antisymmetric on Prop($\mathscr{L}_{S5}$) since it even fails to be antisymmetric on SProp($\mathscr{L}_{S5}$). Take, for instance, $\tau := \Box(p \wedge q)$, $\varphi := \Diamond(p \wedge \neg q)$ and $\psi \equiv \tau \vee \varphi$, then $\psi \leq^{\boxplus}_{\tau} \varphi$, $\varphi \leq^{\boxplus}_{\tau} \psi$, but $\psi \not\equiv \varphi$; therefore $\leq^{\boxplus}_{\tau}$ is not a partial ordering of SProp($\mathscr{L}_{S5}$) (and a fortiori not a partial ordering of Prop($\mathscr{L}_{S5}$)).

Let us consider the upper and lower bounds of the present truth- and falsity-contents. Let φ be an element of Prop($\mathscr{L}_{S5}[p_1, \ldots, p_n]$); then

$$\{\top\} \subseteq Ct^{+}_{T}(\varphi) \subseteq \mathrm{Cn}(\tau), \text{ and}$$
$$\varnothing \subseteq Ct^{\Box}_{F}(\varphi) \subseteq \{w \in \mathfrak{W} \mid a\,R\,w\} = \mathrm{Wrd}^{i}(\tau),$$

The tautology $\top$ and the negation of the truth $\neg\tau$ both have the *minimal truth content*: $\{\top\} = Ct^{+}_{T}(\top) = Ct^{+}_{T}(\neg\tau)$. They also have the *minimal falsity content*: $\varnothing = Ct^{\Box}_{F}(\top) = Ct^{\Box}_{F}(\neg\tau)$. All theories with a larger falsity content, however, also have a larger truth content. Consequently, for all true $\psi \in$ Prop($\mathscr{L}_{S5}$): $\psi \leq^{\boxplus}_{\tau} \neg\tau$ or $\psi \leq^{\boxplus}_{\tau} \top$; using $\leq^{\boxplus}_{\tau}$ we cannot compare any false φ with $\top$ or $\neg\tau$.

As a result of the Cn-clause, all the modal constituents are uncomparable. The *complete falsehood* ξ and the *contradiction* $\perp$, interpreted to be the sentence that excludes all possible worlds, have the *maximal falsity content*:[44] $\mathrm{Wrd}^{i}(\tau) = Ct^{\Box}_{F}(\perp) = Ct^{\Box}_{F}(\xi)$. It depends, however, on the true constituent whether other propositions also have the maximal falsity content. For instance, if in $\mathscr{L}_{S5}[p, q]$, the truth $\tau := \Box(p \wedge q)$ half of the modal constituents have the maximal falsity content. If, however, the truth is: $\Box(p \vee q) \wedge \Diamond(p \wedge q) \wedge \Diamond(p \wedge \neg q) \wedge \Diamond(\neg p \wedge q)$ none except for $\xi := \Box(\neg p \wedge \neg q)$ has the maximal falsity content. More generally, we may say that $\{\gamma \in \mathrm{Cnst}(\mathscr{L}_{S5}) \mid \forall w \in \mathfrak{W}: w \in \mathrm{Wrd}^{e}(\gamma)\}$ is the set of modal constituents with the largest falsity content; it contains the complete falsehood ξ. Note, however, that constituents with maximal falsity-content need not be the worst theories of Prop($\mathscr{L}_{S5}$). Due to the Cn-clause, every disjunction of those constituents is worse; it has the same falsity-content but a smaller truth-content. This observation shows that the definition does *not collapse under the Miller-Tichý argument* as two different false propositions may be comparable. Let us now turn to the more specific properties of the ⊞-definition.

Our first observation concerns the *independence of the consequence and falsity clause*; $Ct^{+}_{T}(\psi) \supset Ct^{+}_{T}(\varphi)$ does not exclude a. $Ct^{\Box}_{F}(\psi) \subset Ct^{\Box}_{F}(\varphi)$ nor b. $Ct^{\Box}_{F}(\psi) \supset Ct^{\Box}_{F}(\varphi)$. Regarding the first combination consider $\tau := \Box(p \wedge q)$, $\varphi := \Box\neg p$, and $\psi := \Box(\neg p \wedge \neg q) \vee \Box(p \wedge q)$. Since $\psi \vDash \varphi \vee \tau$ and $\varphi \nvDash \psi$ the Cn-clause is fulfilled, $Ct^{+}_{T}(\psi) \supset Ct^{+}_{T}(\varphi)$, and all physically possible worlds falsify φ and confirm ψ. This means $Ct_F^{\Box}(\psi) \subset Ct_F^{\Box}(\varphi)$, and $\psi \leq_\tau^{\boxplus} \varphi$. Regarding the second combination I define τ, φ and ψ as follows:

$$\begin{aligned}
\tau &:= \Box(p \leftrightarrow q) \wedge \Diamond(p \wedge q) \wedge \Diamond(\neg p \wedge \neg q)\\
\psi &:= \Box(\neg p) \wedge \Diamond(\neg p \wedge q) \wedge \Diamond(\neg p \wedge \neg q)\\
\varphi &:= \Box(\neg p \vee q) \wedge \Diamond(\neg p \wedge q)
\end{aligned}$$

Then $\psi \vDash \varphi$, $\varphi \vDash \psi \vee \tau$ and $Ct_T^{+}(\psi) \supset Ct_T^{+}(\varphi)$. The physically possible world $\langle 1,1\rangle$, however, falsifies ψ, since ψ rules out all worlds in which p holds; however, $\langle 1,1\rangle$ does not falsify φ, $\langle 1,1\rangle \vDash \varphi$, and $Ct_F^{\Box}(\psi) \supset Ct_F^{\Box}(\varphi)$. Consequently, $\psi \not\leq_\tau^{\boxplus} \varphi$ nor $\varphi \not\leq_\tau^{\boxplus} \psi$.

As befits a genuine content definition, false modal constituents may be better than some of their false consequences. The reverse of this statement, however, is emphatically false; there are antecedences of a false theory ψ that are not closer to the truth than ψ (see preceding example). Although φ is false—it claims that $\neg p \wedge q$ is physically possible whereas it is not—ψ, an arbitrary antecedent of φ, is not as close to the truth as φ. In other words, just as the 2Δ-proposal, my $\boxplus$-definition *blocks* Tichý's *child's-play argument.* On the other hand, the Cn-clause guarantees that if the truth-values of φ and ψ are the same, then $\psi \leq_\tau^{\boxplus} \varphi$ implies $\psi \vDash \varphi$.

We embarked on the development of the $\boxplus$-definition because of its alleged *truth-value independency.* It is easy to show that according to the modal Cn-proposal weak and true propositions can be *less* verisimilar than stronger false ones. Let

$$\tau := \Box(p \wedge q), \quad \psi := \Box p \wedge \Diamond(p \wedge q) \wedge \Diamond(p \wedge \neg q), \quad \varphi := \Diamond p$$

Then, ψ is false and φ is not. Furthermore, the Cn-clause obtains, $Ct_T^{+}(\psi) \supset Ct_T^{+}(\varphi)$, since $\psi \vDash \varphi$ (and $\varphi \nvDash \psi \vee \tau$). Finally, the falsity clause holds trivially as there is no w in $\mathfrak{W}$ such that it is accessible and w falsifies ψ.

Summarizing the preceding observations, we may say that the modal Cn-proposal provides a preordering of the propositions of $\mathcal{L}_{S5}[p,q]$; because of its minimal truth-content, there is no empirical proposition that is worse than the negation of the truth. If the complete falsehood ξ is the sole constituent with the maximum of falsity content, then there is no empirical proposition that is worse than ξ. The $\boxplus$-definition withstands the Miller-Tichý argument. Finally, by taking falsification seriously it blocks the child's-play argument, and it is truth-value independent.

2.7. SUMMARY

Now that we have extensively considered Miller's and Kuipers's content definitions and our modal proposals, let me summarize the present chapter.

1. Miller and Kuipers replace Popper's original definition by the *Δ-distance between sentences*—that is ψ is at least as close to the truth as φ iff $\mathrm{Mod}(\psi)\ \Delta\ \mathrm{Mod}(\tau) \subseteq \mathrm{Mod}(\varphi)\ \Delta\ \mathrm{Mod}(\tau)$.
2. The Δ-proposal *preserves Popper's Cn-clause*, $\mathrm{Cn}(\psi) \supseteq \mathrm{Cn}(\varphi) \cap \mathrm{Cn}(\tau)$, and *replaces his falsity clause* $\mathrm{Cn}(\varphi) \supseteq \mathrm{Cn}(\psi) \cap \mathrm{An}(\neg\tau)$, by the antecedence clause: $\mathrm{An}(\psi) \supseteq \mathrm{An}(\varphi) \cap \mathrm{An}(\tau)$.
3. Replacement of the falsity clause by the An-clause makes the Δ-definition
   a. truth-value dependent—that is, no false proposition is better than a true one
   b. suffer from the *child's-play* argument—every antecedence of a false φ is closer to the truth than φ.
4. The Δ-proposal is a *content definition* as it is based on truth-value and logical strength. It abstains from considering the likeness between the possible worlds; and the negation of the truth is the worst contingent proposition of the language.
5. Using modal intuitions, Kuipers argues in favour of an *incomplete truth*. His proposal, however, leads to the uncomparability of logical strength of empirical theories. Kuipers's modal intuitions are better formalized using modal languages than by the assumption that the truth is incomplete.
6. I reformulated Kuipers's modal intuitions using an *S5-language* and *a complete truth*. This led to two modal content definitions.
   a. The *2Δ-definition*, which restores the falsity clause. It blocks the child's-play argument and solves the problems concerning the logical strength of theories; however, it inherits the truth-value dependency of the Δ-definition.
   b. The ⊞-definition preserves all advantages of the 2Δ-definition and even solves the truth-value dependency problem. It prepares the way for combining my proposal presented in Chapter 6 with the modal approach.

CHAPTER 3

TRUTHLIKENESS

We encountered the distinction between verisimilitude and truthlikeness definitions in Chapter 1. Verisimilitude definitions define distance to the truth using truth-value and logical strength, and were presented in Chapter 2. Truthlikeness definitions establish distance to the truth using similarity between the possible worlds, and are the subject of the present chapter.

Truthlikeness definitions originate from two different sources. The first root is Hilpinen's address to the Warsaw conference in 1974.[1] In Section 3.1, we shall see how Hilpinen uses D. Lewis's (1973) counterfactuals to specify his ideas about an approximate logic and truthlikeness.[2] Subsequently, it was Niiniluoto who took up Hilpinen's comparative approach, and converted it into a quantitative truthlikeness proposal. His approach is the subject of Section 3.2. Kuipers published his first version of comparative truthlikeness in 1987. His refined comparative definition is the subject of Section 3.5.

Pavel Tichý's 1974 paper forms the second root of the truthlikeness strategy. In this publication, he revealed the major flaw of Popper's original proposal, and initiated a new truthlikeness approach based on Hintikka's constituents. Oddie adjusted and elaborated Tichý's quantitative method; this forms the subject of the Section 3.3.[3] More recently, Heidema in cooperation with Brink and Burger have published qualitative truthlikeness definitions.[4] They order the propositions of a two-propositional language (almost) in the same way as Tichý and Oddie. Hence, as the quantitative-comparative distinction seems to be subsidiary, and I preferred a historical line of presentation to the systematic one, I shall introduce the proposal of Heidema and others in Section 3.4. The last section concerns the conclusions of this chapter.

3.1. HILPINEN

Hilpinen's *Approximate Truth and Truthlikeness* includes the main ingredients of his address to the Warsaw conference in 1974. It is remarkable in two respects. In the first place, the conference was held before Tichy's and Miller's commentary on the flaw in Popper's definition was published and generally acknowledged. In the second place, it was the first publication that stressed the difference between likeness and content approaches, and introduced the *truthlikeness* notion. Our introduction summarizes Hilpinen (1976).

3.1.1. *The Definition*

Hilpinen fixes the similarity between theories, and more specifically the similarity between a theory and the truth, by setting up an approximate *A-logic*. This approximate logic is the standard propositional logic enriched with a new approximate operator **A**. The basic idea is that a statement *'P'* is *approximately true* at world u, if and only if, P is true in some world similar (or close to) to u. U is the class of all possible worlds of the language. Suppose that N_u refers to the set of possible worlds close (or similar) to u, and let P be a proposition of the language, then,

$$\vDash_u \mathbf{A}P \text{ if and only if } \vDash_v P \text{ for some } v \in N_u$$

No N_u is empty since $u \in N_u$, and the similarity relation is symmetric. The A-logic turns out to be at least as strong as the Brouwerian system of modal logic.[5]

Next, Hilpinen chooses *characteristics* C^i fixing the mutual likeness of the possible worlds. It is important to note that Hilpinen *is the only one* who explicitly mentions the relation between the *independently* chosen characteristics C^i and the likeness between possible worlds. This will prove to be the most important part of the solution to the problem of alleged 'language-dependency' of likeness definitions.[6] Following David Lewis, Hilpinen defines a *system of nested spheres* $\mathcal{N}^i_u$ related to a world $u \in U$ and to a characteristic C^i. It is a family of subsets of U such that

For every $K, L \in \mathcal{N}^i_u$, $K \subseteq L$ or $L \subseteq K$ (nesting), and
$u \in K$ for all $K \in \mathcal{N}^i_u$ (weak centering).

Consequently, $u \in U$, $K \in \wp(U)$, and $\mathcal{N}^i_u \in \wp(\wp(U))$. Hilpinen assumes that $\mathcal{N}^i_u$ is closed under taking (nonempty) intersections and unions, and therefore that $\cap\mathcal{N}^i_u$ is the smallest, and $\cup\mathcal{N}^i_u$ is the largest sphere around u. Mod($\mathcal{L}$) - $\cup\mathcal{N}^i_u \neq \varnothing$ if there are worlds not similar to u at all. To simplify the comparison with the other definitions, I assume that, in addition to being centering, $\mathcal{N}^i_u$ *is centered*—i.e. $\{u\} \in \mathcal{N}^i_u$. Furthermore, from now on, I assume that C^i is fixed, and shall drop the index i. Nevertheless, Hilpinen's truthlikeness definition remains depending on the independent choice of the specific set C^i of characteristics.

Hilpinen's truthlikeness proposal combines two elements. The first one has to do with the *distance* between the truth u and the most u-similar possible world of the theory. The second element concerns the amount of *information of the theory*, or its 'verbosity'. Informally, the definition of Hilpinen says that the theory P has a higher degree of truthlikeness than Q if and only if the minimum distance of P to the truth is smaller than the distance of Q to the truth, and P is more informative about the truth than Q. Hilpinen uses the systems of nested spheres to give an exact definition of $E_u(P)$, the distance between P and u, and of $I(P)$, the amount of information of P. He defines $E_u(P)$, by way of the set of elements of $\mathcal{N}_u$ that do not contain models for P:

(1) $\quad E_u(P) :=_{def} \{K \mid K \in \mathcal{N}_u \text{ and } K \cap \mathrm{Mod}(P) = \varnothing\}$

$E_u(P)$ is the set of spheres around u that do not intersect with Mod(P). Thus, $E_u(P)$ is the set of spheres "between" $\cap\mathcal{N}_u$ and Mod(P) (see Figure 1). The definition of $E_u(P)$ yields the following propositions.

$E_u(\top) = \varnothing$
$E_u(\bot) = \mathcal{N}_u$
$E_u(P \vee Q) = E_u(P) \cap E_u(Q)$
$E_u(P) \subseteq E_u(P \wedge Q)$
$E_u(P) = \varnothing$ or $E_u(\neg P) = \varnothing$

Hilpinen also uses his system of nested spheres to define the amount of information of hypothesis P. The more distant worlds P excludes, the more information P provides about the truth u. Formally, he defines:

(2) $\quad I_u(P) :=_{def} \{K \mid K \in \mathcal{N}_u \text{ and } \varnothing \neq \mathrm{Mod}(P) \subseteq K\}$

If the definition had allowed Mod(P) to be empty, then $P = \bot$ would have implied $I_u(\bot) = \mathcal{N}_u$, and the contradiction would have been maximally informative about the truth. Hilpinen avoids this counter-intuitive consequence, with the constraint Mod(P) $\neq \varnothing$; hence the contradiction is the worst element. Intuitively, $I_u(P)$ is the set of spheres 'between' Mod(P) and the largest sphere $\cup\mathcal{N}_u^i$. Hilpinen"s definition implies the following propositions.

$I_u(\top) = \varnothing$ (in case there are uncomparable worlds)
$I_u(\bot) = \varnothing$
if Mod(P) $\neq \varnothing$ and Mod(Q) $\neq \varnothing$, then: $I_u(P \vee Q) = I_u(P) \cap I_u(Q)$
if Mod($P \wedge Q$) $\neq \varnothing$, then: $I_u(P) \subseteq I(P \wedge Q)$
$I_u(P) = \varnothing$ or $I_u(\neg P) = \varnothing$

Hilpinen defines truthlikeness by combing $E_u(P)$ and $I_u(P)$. Let P and Q be propositions of an A-logic $\mathcal{L}$, and let model u (or τ) represent the true model. Then,

DEFINITION 3.1: The *truthlikeness of Q does not exceed that of P* if and only if $E_u(P) \subseteq E_u(Q)$ and $I_u(Q) \subseteq I_u(P)$
Notation: $P \leq_u^H Q$ (or $\psi \leq_\tau^H \varphi$)

Since $\leq_u^H$ is reflexive and transitive, but not antisymmetric, it is a preordering of the $\mathcal{L}$-propositions. Intuitively, definition 3.1 boils down to: Mod(P) is at least as close to u as Mod(Q) is, and, by excluding more distant worlds, Mod(P) reveals at least as much information about u as Mod(Q). Figure 1 depicts this situation.

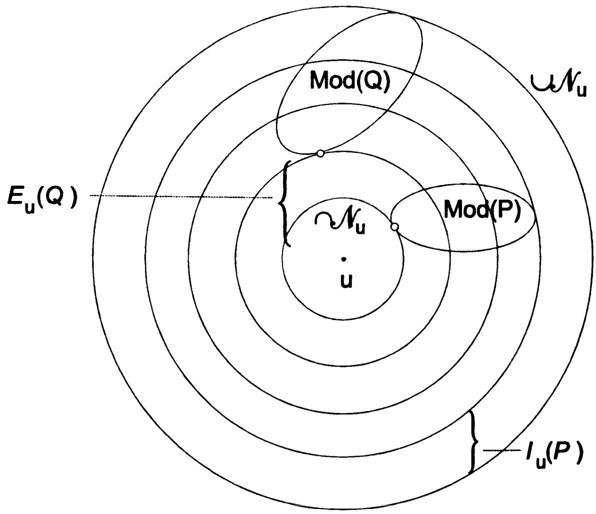


Fig. 1. The nested spheres $\mathbf{N}_u$

Note that, strictly speaking Hilpinen orders *sentences*, such as P and Q, by their similarity to a *possible world u*. No other proposal presented here defines similarity between sentences on the one hand, and models on the other. In addition to definition 3.1, Hilpinen defines (I use my symbols):

$P <_u Q :=_{def} P \leq_u Q$ and $Q \nleq_u P$, P is more truthlike than Q
$P =_u Q :=_{def} P \leq_u Q$ and $Q \leq_u P$, P and Q are equal truthlike;
$P \parallel Q :=_{def} P \nleq_u Q$ and $Q \nleq_u P$, P and Q are uncomparable regarding u (and C^i).

Two theories P and Q are uncomparable if the distance factor and the information factor point in different directions.[7] This obtains if P is closer to the truth but Q is more informative, or the reverse is true:

$E(P) \subset(\supset) E(Q)$ and $I(P) \subset(\supset) I(Q)$

Finally, we show how Hilpinen's approach fares in the bi propositional case. Let $K_1 := \{\langle 1,1\rangle\}$, $K_2 := \{\langle 1,1\rangle, \langle 1,0\rangle, \langle 0,1\rangle\}$ and $K_3 :=$ Mod($\mathcal{L}$) and $\mathcal{N}_u := \{K_1, K_2, K_3\}$. Since $P =_{p\wedge q} Q$ does not imply that P and Q are logically equivalent, we order sets of $\mathcal{L}[p,q]$ propositions.

Figure 2 shows the $\leq^H_{p\wedge q}$-ordering of the sets of $\mathcal{L}[p,q]$-propositions with the same truthlikeness (the truth is $p \wedge q$). It demonstrates, at least in this elementary case, that Hilpinen's definition is truth-value dependent. There is even no false theory that is more similar to $p \wedge q$ than the tautology. Let us turn to some remarks regarding Hilpinen's proposals and examine how it fares regarding the metatheoretical properties mentioned in the first chapter.

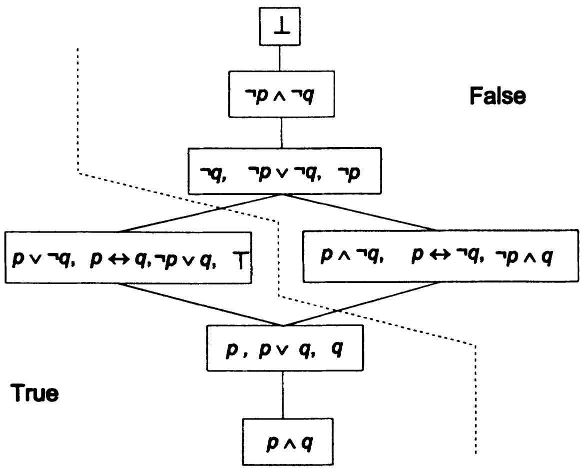


Fig. 2. Hilpinen's ordering of Prop($\mathcal{L}$[p,q])

3.1.2. Remarks and Comments

In the first part of this subsection, I shall characterize the difference between the content approach and Hilpinen's likeness strategy regarding propositional languages. In the second part, we shall discuss the metatheoretical properties of the approximate logic definition of truthlikeness.

The following example illustrates the difference between Hilpinen's truthlikeness and Miller's content definition (the truth is complete in both cases). As Miller's definition is weaker than Hilpinen's proposal, the best way to compare them is to see under which conditions $\psi <_{\tau}^{\Delta} \varphi$ does not imply $\psi \leq_{\tau}^{H} \varphi$. Suppose that $\psi <_{\tau}^{\Delta} \varphi$; then there are four situations to be considered in which Hilpinen's definition establishes a strict ordering ($\mathfrak{W}$ represents the relevant structure of our world):

1. φ and ψ are true; then $E_{\mathfrak{W}}(\varphi) = E_{\mathfrak{W}}(\psi) = \varnothing$; (and $I(\varphi) \subset I(\psi)$)
2. ψ is true and φ is false; $E_{\mathfrak{W}}(\varphi) \supset E_{\mathfrak{W}}(\psi)$; (and $I(\varphi) \subseteq I(\psi)$)
3. ψ and φ are false and $E_{\mathfrak{W}}(\varphi) = E_{\mathfrak{W}}(\psi)$; (and $I(\varphi) \subset I(\psi)$)
4. ψ and φ are false and $E_{\mathfrak{W}}(\varphi) \subset E_{\mathfrak{W}}(\psi)$; (and $I(\varphi) \subseteq I(\psi)$)

The nested spheres around the constituent $p \wedge q \wedge r$ of a language $\mathscr{L}[p,q,r]$, and the four situations just mentioned are depicted in Figure 3. Although in situations I, II and III, Miller's ordering implies the one of Hilpinen, situation IV blocks $\psi <^{\Delta}_{\tau} \varphi \Rightarrow \psi \leq^{H}_{\tau} \varphi$. In case four, although $E_{\mathfrak{W}}(\varphi) \subset E_{\mathfrak{W}}(\psi)$, Miller prefers the stronger false proposition ψ, since it has more true consequences. According to Hilpinen, φ is closer to the truth than ψ; regarding the minimal distances φ is closer to $\mathfrak{W}$ than ψ, and $\psi <^{\Delta}_{\tau} \varphi$ guarantees that the maximum distance between $\mathfrak{W}$ and φ is at least as large as that between $\mathfrak{W}$ and φ. Finally, if the maximum distance to $\mathfrak{W}$ of the weakest is also larger, the propositions are uncomparable regarding their truthlikeness. This example epitomizes the general case; if φ and ψ are false, Miller's and Hilpinen's proposals contradict if $\psi \vDash \varphi$ and $E_{\mathfrak{W}}(\varphi) \subset E_{\mathfrak{W}}(\psi)$ and $I(\varphi) = I(\psi)$. Thus, this most elementary case already shows the fundamental difference between the content and the likeness strategy. The former only regard logical strength and truth-value, whereas the latter is based on similarity between possible worlds.

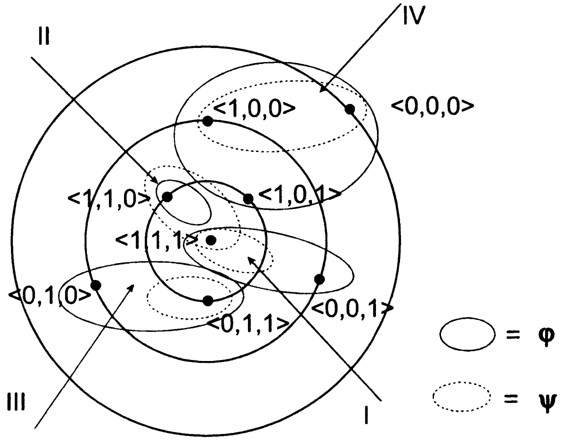


Fig. 3. $\varphi <^{\Delta}_{\tau} \psi$ and $\psi \leq^{H}_{\tau} \varphi$

The example also shows that Hilpinen's definition is a combination of minimal and *maximal* distances rather than a combination of minimal distances and *content*. On the one hand, propositions with the same truthlikeness might differ regarding logical strength. For instance although, $((p \wedge q \wedge r) \vee (\neg p \wedge \neg q \wedge \neg r)) \vDash p \rightarrow q \vee r$ and the inverse does not hold, according to Hilpinen's definition, the propositions are equally truthlike. On the other hand, $\psi =^{H}_{\tau} \varphi$ obtains if and only if $I_{\tau}(\varphi) = I_{\tau}(\psi)$, and $E_{\tau}(\varphi) = E_{\tau}(\psi)$, which says that the maximal distances and the minimal distances between φ/ψ and the truth are the same.[8]

Let us turn to the examination of the metatheoretical properties of Hilpinen's proposal. Hilpinen's definition is a *comparative* proposal, which does not use quantitative methods. It is truth-value dependent. False propositions cannot be better than true ones, since for true propositions ψ, $E_\tau(\psi)$ is the empty set, whereas for false propositions φ, $E_\tau(\varphi)$ is nonempty.[9] Furthermore, according to Hilpinen's definition, the *complete falsehood* is the least truthlike element of EmProp($\mathcal{L}[p_1, ..., p_n]$).[10]

PROPOSITION 3.1: The *complete falsehood* ξ ($:= \neg p_1 \wedge ... \wedge \neg p_n$) is the *greatest element* of EmProp($\mathcal{L}[p_1, ..., p_n]$) if ordered by $\leq^H_\tau$, $\tau := p_1 \wedge ... \wedge p_n$ and $C^i := (p_1, ..., p_n)$.

Proof: To prove the proposition we have to show that for all $\psi \in$ EmProp($\mathcal{L}[p_1, ..., p_n]$) $\psi \leq^H_\tau \xi$ with $\tau := p_1 \wedge ... \wedge p_n$; observe that $E_\tau(\xi) :=$ Mod($\mathcal{L}$)$- \{\langle 0, ..., 0\rangle\}$ and $I_\tau(\xi) = \varnothing$. Therefore, for all $\psi \in$ EmProp($\mathcal{L}$) we must prove (by definition) $E_\tau(\psi) \subseteq$ Mod($\mathcal{L}$)$- \{\langle 0, ..., 0\rangle\}$ and $\varnothing \subseteq I_\tau(\varphi)$. As the latter holds trivially, only if $\psi \equiv \perp$ or if $\psi \equiv \xi$ the condition fails; therefore for all $\psi \in$ EmProp($\mathcal{L}$), $\psi \leq^H_\tau \xi$. ⊠

As to specularity, it suffices to observe that Hilpinen's definition assumes that the truth is complete therefore specularity does not apply. Let us assume that a more elaborate version of the definition also covers the case in which the truth is not complete. Even then, because the minimal and maximal elements of the $\leq^H_\tau$-ordering are of equal logical strength, it does not seem likely that an indefinite truth version of the definition will be specular.

Let us turn to the language dynamics of Hilpinen's definition. The next observation needs the notion of a conservative extension of the set of characteristic C. I shall say that C is conservatively extended to C' if independent and relevant characteristics c_i are added to C. Let $\mathcal{N}_\tau$ be a set of nested spheres around Mod(τ) in Mod($\mathcal{L}$) and let $\mathcal{N}_\tau$ be based on the characteristics C.

Observation 3.2: If C can be conservatively extended into C' then for the resulting $\mathcal{N}_\tau'$ obtains: $\forall K \in \mathcal{N}_\tau$: $K \in \mathcal{N}_\tau'$.

The reason is clear. Since all old characteristics remain in place, the extension of C can only result in a more fine grained subdivision of $\mathcal{N}_\tau$.

The next proposition about context independence (p. 32) uses the following terminology. $\mathcal{L}$ is some language with complete truth τ, and $\mathcal{N}_\tau$ is a set of nested spheres around Mod(τ) the similarity of which is based on the characteristics C. Let $\mathcal{L}'$ be a conservative extension of $\mathcal{L}$ with complete τ' and let $\mathcal{N}_{\tau'}$ be the set of nested spheres of the models of $\mathcal{L}'$ based on $C' \supset C$. Let σ be the translation from $\mathcal{L}$ to $\mathcal{L}'$, such that for all $\mathcal{L}$-propositions φ, the $\mathcal{L}'$-translation $\sigma(\varphi)$ has the same meaning as φ.

PROPOSITION 3.3: $<^H_\tau$ *is strongly context independent.*

Proof: Let ψ, φ be propositions of Prop($\mathcal{L}$) such that $\psi <^H_\tau \varphi$, and let $\sigma(\psi) = \psi'$, $\sigma(\varphi) = \varphi' \in$ Prop($\mathcal{L}'$) be their translations in $\mathcal{L}' \supseteq \mathcal{L}$. We must prove that $\psi <_\tau \varphi$ implies $\psi' <_{\tau'} \varphi'$. Let us first assume that (*) $E_\tau(\psi) \subset E_\tau(\varphi)$ and $I_\tau(\psi) \supset I_\tau(\varphi)$.

Then, observation 3.2 shows that $\mathcal{N}_{\tau'}$ respects $\mathcal{N}_{\tau}$ in Mod($\mathcal{L}'$). Thus, (*) implies, $E_{\tau'}(\psi') \subset E_{\tau'}(\varphi')$ and $I_{\tau}'(\psi) \supset I_{\tau}'(\varphi)$. ⊠

3.2. NIINILUOTO

Niiniluoto's *Truthlikeness* is the most comprehensive contribution to the approach-to-the-truth research thus far. First, I shall summarize the book, which comprises more than five hundred pages of high information density. The author compared his book with a 'tool box', since different cognitive problems ask for different solutions to the question about truthlikeness. Without thorough study, however, it is not obvious where to find the appropriate tools.

The thirteen chapters of the book are divided into four groups. Chapters one to four deal with the logical and mathematical preliminaries needed to formulate the main proposals of the book. The second group of three chapters contains the top down presentation of Niiniluoto's conception of truthlikeness. The fifth chapter contains an intuitive and philosophical presentation of the main ideas about truthlikeness. The sixth chapter, is the most central and important one. Here, Niiniluoto gives a top down presentation of his definition culminating in a recommendation of a *min-sum-measure* $M_{ms}^{\gamma\gamma'}$ which I shall explain in due course. The group of top down chapters closes with chapter seven, which shows how Niiniluoto tackles the epistemological problem of approach to the truth. In his solution, which is the subject of Chapter 4, he applies Hintikka's λ-α system and uses truthlikeness as an epistemic utility.

After the general introduction of his truthlikeness definition, in the third part, Niiniluoto elaborates special cases in which he applies his general definition. Chapters eight to ten are characterized by their bottom up approach. Chapter eight deals with 'singular statements' covering state descriptions, structure descriptions, and quantitative singular statements. Chapter nine deals with 'monadic generalizations'. In a way this is also an important chapter, since it illustrates how the general definition handles the most elementary cases such as: the monadic constituents with and without identity, and the existential and universal generalisations. The third part closes with a chapter on polyadic theories. The last group of chapters is dedicated to the relation between Niiniluoto's approach and related projects such as L.J. Cohen's legisimilitude proposal in chapter eleven, and cognitive decision making as advocated by I. Levi. In the last chapter, Niiniluoto deals with all kinds of general objections to the truthlikeness project. Obviously, limitations of space force us to present only the barest outline of *Truthlikeness*. In subsection 3.2.1, I present Niiniluoto's general truthlikeness definition, and discuss Niiniluoto's ordering of propositions in subsection 3.2.2.

3.2.1. General Scheme

In this subsection, I introduce Niiniluoto's general truthlikeness definition in six steps. To begin with, Niiniluoto puts up a general framework, or *problem-set* that copes with a variety of situations. This problem set, or *P*-set is related to the background knowledge b.[11] If there is no relevant background information, b is the tautology. The elements of the problem-set are jointly exhaustive, mutually exclusive, and consistent relative to b.

DEFINITION 3.2: $\mathbf{B}$ is a *P*-set relative to b if and only if

$\mathbf{B} = \{h_i \mid i \in \mathbf{I}\}$
$b \vdash \bigvee_{i \in I} h_i$
$b \vdash \neg(h_i \wedge h_j)$ for all $i \neq j$, $i, j \in \mathbf{I}$
not $(b \vdash \neg h_i)$ for all $i \in \mathbf{I}$

$\mathbf{I}$ is an index set ranging over all elements of $\mathbf{B}$. A paradigmatic example of a problem-set is the set of all *constituents* of a finite language $\mathcal{L}$. Constituents are jointly exhaustive, and mutually exclusive; and they are consistent relative to the tautology. If the language $\mathcal{L}$ is determinate, one of the constituents is factually true, and all the others are false and the question: "which element of $\mathbf{B}$ is true?" has a definite and precise answer. Another important example concerns the approximation of the value of some unknown *parameter* θ. $\mathbf{B}$ fixes the upper and the lower bound of the possible values of θ, and again, one value in $\mathbf{B}$ is the true value θ^*. θ must have one value, and the background knowledge b is consistent with any determinate value of θ. Niiniluoto gives many other possible instantiations of $\mathbf{B}$, some of which can be found in Chapter 1.

The second step of the general truthlikeness definition consists of setting up the class of *disjunctive closures* of $\mathbf{B}$. Every $h_i \in \mathbf{B}$ is a complete potential answer to the question: "Which element of $\mathbf{B}$ is true?"; however, not all answers to a question need to be complete. Incomplete or partial potential answers also are relevant for a cognitive problem, and subsets of $\mathbf{B}$-elements represent incomplete answers. To assess these partial answers, Niiniluoto defines the disjunctive closure $D(\mathbf{B})$ of $\mathbf{B}$: $D(\mathbf{B}) :=_{def} \{\bigvee_{i \in J} h_i \mid \varnothing \neq J \subseteq I\}$. Every partial answer g has a $\mathbf{B}$-normal form: $g := \bigvee_{j \in Ig} h_j$.

The third step is the specification of the requirements for the *distance function* Δ: $\mathbf{B} \times \mathbf{B} \rightarrow \mathbb{R}$ on the problem space. This distance $\Delta_{ij} = \Delta(h_i, h_j)$ between the elements h_i and h_j of $\mathbf{B}$ is at least a *semimetric*:

$\Delta_{ij} \geq 0$ for all $i,j \in I$
$\Delta_{ij} = 0$ if and only if $i = j$

Often, Niiniluoto uses a *normalized* distance function such that: $\Delta_{ij} \leq 1$, and sometimes the distance function is even a *metric*; then, it complies with

$\Delta_{ij} = \Delta_{ji}$ for all $i,j \in I$
$\Delta_{ij} \leq \Delta_{ik} + \Delta_{kj}$ for all $i, j, k \in I$

Niiniluoto calls distance Δ_{ij} *balanced* if $|I|^{-1} \sum_{j\in I}(\Delta_{ij}) = ½$ for all $i \in I$.

These constraints leave room for many distance functions on a problem space, and Niiniluoto studies a variety of them. For instance, if $\Theta \subseteq \mathbb{R}$ is an interval of the real numbers, then all $\theta \in \Theta$ are complete potential answers to the cognitive problem "which θ of Θ has the true value θ^*?" Then, Niiniluoto uses the metric,

$$\Delta(x,y) := |x-y| \quad \text{for } x, y \in \mathbb{R}.$$

Another example concerns constituents C_i of a monadic language $\mathcal{L}_N^k$ claiming that the set $\mathbf{CT}_i$ of Carnapian Q-predicates is exemplified and its complement is empty. Niiniluoto uses the cardinality of the symmetric difference between $\mathbf{CT}_i$ and $\mathbf{CT}_j$ as a distance function on a cognitive problem of constituents:[12]

$$\mathrm{d}(\mathbf{CT}_i, \mathbf{CT}_j) := |\mathbf{CT}_i \,\Delta\, \mathbf{CT}_j|$$

He calls the normalized version of this function the *Clifford measure*:[13]

$$\mathrm{d}_C(\mathbf{CT}_i, \mathbf{CT}_j) := \frac{1}{q}\,|\mathbf{CT}_i \,\Delta\, \mathbf{CT}_j|$$

in which q := number of possible constituents. If k is the number of atomic predicates in $\mathcal{L}_N^k$ then $K = 2^k$ is the number of Q-predicates and $q = 2^K - 1$.[14]

Now that the problem space **B**, the disjunctive closure D(**B**), and the distance Δ_{ij} on **B** are defined, the fourth step is *providing the disjunctive closure with an extension* Δ: $\mathbf{B} \times D(\mathbf{B}) \rightarrow \mathbb{R}$ of the distance function Δ_{ij}. This function has to be a natural extension of the distance function Δ_{ij} on **B**. In other words, Δ on $\mathbf{B} \times D(\mathbf{B})$ must naturally reduce to the distance function Δ_{ij} on **B**. Niiniluoto calls this measure on the disjunctive closure the *reduction function*, red:[15]

$$\Delta(h_i,g) = \mathrm{red}(\langle \Delta_{ij} \mid j \in \mathbf{I}_g \rangle).^{16}$$

A natural adequacy condition for discrete problem spaces is that the reduction function should preserve the ordering of the problem space **B**. Assume that $\mathbf{I}_g = \{j\}$ and $\mathbf{I}_{g'} = \{k\}$. Then this adequacy condition says $\Delta(h_i,g) \leq \Delta(h_i,g')$ if and only if $\Delta_{ij} \leq \Delta_{ik}$[17]. This condition is fulfilled if the Δ-function value of $h_i \in \mathbf{B}$ and a $\{h_j\} \in D(\mathbf{B})$ is equal to the distance between h_i and h_j—that is $\mathrm{red}(\langle \Delta_{ij} \rangle) := \Delta_{ij}$.

Defining the truthlikeness of an answer in terms of closeness to the truth is the fifth step in our presentation. Let h^* be the truth in some problem space **B**, let Δ_{ij} be the distance on **B**, and let $\Delta(h^*,g) = \mathrm{red}(\langle \Delta_{*j} \mid j \in \mathbf{I}_g \rangle)$. If Δ is normalised, Niiniluoto defines the *closeness of* $g \in D(\mathbf{B})$ to the truth h^* by:

$$M(g,h^*) :=_{def} 1 - \Delta(h^*,g)$$

Suppose **B** is a problem set, Δ_{ij} is a distance function on **B**, and Δ: $\mathbf{B} \times D(\mathbf{B}) \to [0,1]$ is based on Δ_{ij}. Let the partial answers g', $g \in D(\mathbf{B})$, and $h^* \in \mathbf{B}$ and let $M(g', h^*) := 1 - \Delta(h^*, g')$. Then, Niiniluoto defines truthlikeness as follows.[18]

DEFINITION 3.3: g' is *at least as close to the truth* h^* as g iff
$M(g', h^*) \geq M(g, h^*)$
Notation: $g' \leq^{Ni}_{h^*} g$

The irreflexive version of the definition reads: $g' <^{Ni}_{h^*} g$ iff $M(g', h^*) > M(g, h^*)$.

The last, and sixth step of our introduction is the presentation of specific reduction functions. The distance function on **B** can be extended in many ways to the reduction function of the disjunctive closure of **B**. If the problem space is finite, important options are:[19]

$$\Delta_{\min}(h_i, g) := \min_{j \in \mathbf{I}_g} \Delta_{ij}$$

$$\Delta_{\max}(h_i, g) := \max_{j \in \mathbf{I}_g} \Delta_{ij}$$

$$\Delta_{\mathrm{av}}(h_i, g) := \frac{1}{|\mathbf{I}_g|} \sum_{j \in \mathbf{I}_g} \Delta_{ij}$$

$$\Delta_{\mathrm{sum}}(h_i, g) := \frac{\sum_{j \in \mathbf{I}_g} \Delta_{ij}}{\sum_{j \in \mathbf{I}} \Delta_{ij}} = \frac{\sum_{j \in \mathbf{I}_g} \Delta_{ij}}{|\boldsymbol{I}|av(i, \boldsymbol{B})}$$

$$\Delta_{\mathrm{sum}}(h_i, g) := \frac{2}{|\boldsymbol{I}|} \sum_{j \in \mathbf{I}_g} \Delta_{ij} \quad (\textit{if } \boldsymbol{B} \textit{ is balanced})$$

$$\Delta^{\gamma\gamma'}_{ms} := \gamma \Delta_{\min}(h_i, g) + \gamma' \Delta_{\mathrm{sum}}(h_i, g) \qquad (\gamma > 0, \gamma' > 0)$$

Figure 4 illustrates the mechanism of some reduction functions.[20] It shows a problem set **B**—or rather the index set of **B**—and all the h_j at the same distance to h_i encircle h_i, which resides in the centre. $\mathbf{I}_g$ is the index set belonging to the partial potential answer g. The $\Delta_{\min}(h_i, g)$, and the $\Delta_{\max}(h_i, g)$ do not need further comment. The arithmetical mean, $\Delta_{\mathrm{av}}(h_i, g)$, depends on the size of g and the minimal distant between g and h_i; and the normalized sum, $\Delta_{\mathrm{sum}}(h_i, g)$ reflects both the size of g and its location relative to h_i ($|I|\mathrm{av}(i, \mathbf{B}) := \sum_{j \in I} \Delta_{ij}$). Comparing Figure 3 with Figure 4, we see that Niiniluoto's definition is a quantitative version of Hilpinen's comparative truthlikeness definition. Similar to Hilpinen's proposal, improvement of g consists in getting rid of bad elements of g, or adding new elements to g that improve even the best of g. Of course, Niiniluoto's quantitative version allows more sophisticated refinements. For instance, it may avoid the truth-value dependence of Hilpinen's definition.

Niiniluoto favours the *min-sum measure* $M_{ms}^{\gamma\gamma'}$, which is a weighted sum of $\Delta_{\min}$ and Δ_{sum}. The first is the minimum distance between g and the truth. If $\Delta_{\min}(h_i,g)$ is sufficiently small, g is almost true, or approximately true. Answer g is true if the minimum distance is zero. The second factor, the sum-function Δ_{sum}, reflects the 'size' of the answer g in relation to the average distance to the truth. Roughly, 'small' and logically strong answers are better than 'large' and weak answers that exclude fewer logical possibilities. Apparently, Niiniluoto's quantitative likeness approach also contains considerations of content.

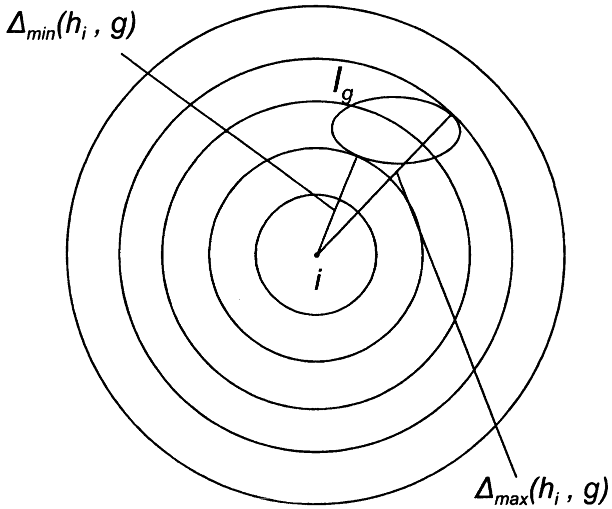


Fig. 4. Equidistant h_i of **B**, *and set* I_g *of* g.

The sum-function is an indicator of verbiage, for answers with the same average distance to h_i, It adds all the distances between the g-members and the truth, and divides it by the sum of the distances between all elements of **B** and the truth. If the distance Δ on **B** is balanced, then the sum-function is equal to $2|\mathrm{I}|^{-1}\Sigma_{j\in \mathrm{I}_g}\Delta_{ij}$.

An important difference between the min-sum reduction function $\Delta_{ms}^{\gamma\gamma'}$ and the other functions is that it uses parameters. These parameters γ and γ' are weights to be adjusted to conditions of a specific context. For instance, if, in particular circumstances, truth is more important than excluding falsity, γ, the weight for the $\Delta_{\min}(h_i,g)$, must be larger than γ', the weight of $\Delta_{\text{sum}}(h_i,g)$. Thus, Niiniluoto does not propose one truthlikeness definition. He proposes a truthlikeness *scheme*, which has to be adjusted to the particularities of the circumstances of the formal application.

3.2.2. The Propositional Case

In this subsection, we shall see how the definition orders the propositions of a finite propositional language. Let $\mathscr{L}[p_1,...,p_n]$ be a propositional language with vocabulary voc($\mathscr{L}$) := $\{p_1,...,p_n\}$. A $\mathscr{L}$-constituent describes one $\mathscr{L}$-model; it has the following form:

$$C_i := (\pm)p_1 \wedge (\pm)p_2 \wedge ... \wedge (\pm)p_i \wedge ... \wedge (\pm)p_n$$

in which $(\pm)p_i$ are literals. An arbitrary proposition of $\mathscr{L}$ can be presented by a disjunction of several constituents (see Chapter 1). For all $\sigma \in$ Prop($\mathscr{L}$) holds $\vdash \sigma \equiv \bigvee_{i \in \mathbf{I}\sigma} C_i$. If **I** is the index set of all constituents of $\mathscr{L}$ then $\mathbf{I}_\sigma \subseteq \mathbf{I}$, is the index set of σ.

Niiniluoto's *Truthlikeness* does not explicitly deal with propositional languages. I shall treat them as Niiniluoto (1987) treats singular statements in his chapter eight. There, the author proposes to take the *mean character difference* as distance function if the cognitive problem consists of singular statements.[21] He also mentions Sokal's *taxonomic* distance, which is the square root of the mean character distance.[22] This does not affect our conclusions, but illustrates the absence of compelling arguments for preferring one distance function to another.

As an alternative to the disjunctive normal form, a constituent can also be represented by an n-tuple of ones and zeros representing the affirmation and negation of corresponding atomic propositions. If we take p_i^j to be a zero or one belonging to constituent j on place i, then the mean character difference is defined by:

$$\frac{1}{n}\sum_{i=1}^{n} |p_i^j - p_i^k|$$

For example, for $n = 4$, $C_j := \neg p_1 \wedge p_2 \wedge \neg p_3 \wedge \neg p_4$, and $C_k := \neg p_1 \wedge p_2 \wedge \neg p_3 \wedge p_4$ correspond to $\langle 0,1,0,0\rangle$ and $\langle 0,1,0,1\rangle$, respectively. Then, $\Delta(C_j, C_k) = ¼$ and the likeness of C_k to C_j is ¾. Note that, if we let Q-predicates correspond with literals, the mean character difference is equivalent to the Clifford measure mentioned in the preceding subsection. The next observation claims that the mean character difference Δ_{jk} on the set of $\mathscr{L}$-constituents is *balanced.*

Observation 3.4: $\dfrac{1}{|\mathbf{I}|}\sum_{k \in \mathbf{I}} \Delta_{jk} = ½$ for all $j \in \mathbf{I}$

Proof: The sum of all distances of all constituents C_k to some arbitrary constituent C_j is equal to $\sum_{i=1}^{n} \binom{n}{i}\, {}^{i}/_{n}$, where n is the number of atomic propositions, and i the number of atomic propositions on which the constituents disagree. Consequently,

$\sum_{k\in\mathbf{I}}\Delta_{jk} = \sum_{i=1}^{n} \binom{n}{i}{}^{i}/_{n}$. Now, $\sum_{i=1}^{n} {}^{n!}/_{i!(n-i)!}{}^{i}/_{n} = \sum_{i=1}^{n} {}^{(n-1)!}/_{(i-1)!(n-i)!} = \sum_{i=1}^{n} \binom{n-1}{i-1}$ $= \sum_{j=0}^{n-1} \binom{n-1}{j} = 2^{n-1}$ and thus ${}^{1}/_{|\mathrm{I}|}\sum_{k\in\mathbf{I}}\Delta_{jk} = 2^{-n}.2^{n-1} = ½$ ⊠

The set of constituents forms a problem set **B**, because they are mutually exclusive and jointly exhaustive. Additionally, to apply Niiniluoto's truthlikeness definition, I choose $\Delta_{ms}^{\gamma\gamma'}(h_i,g)$ as a reduction function. The combination of the Clifford measure and the min-sum reduction function results in the following formula fixing the distance between an arbitrary proposition g of $\mathscr{L}$, and the true $\mathscr{L}$-constituent (min(g) is a constituent with the minimal distance between g and h^*):

$$(5) \qquad \Delta_{ms}^{\gamma\gamma'}(h*,g) := \gamma\frac{1}{n}\sum_{i=1}^{n}|p_i^{\,*}-p_i^{min(g)}| + \frac{1}{2^{n-1}}\gamma'\sum_{j\in I_g}\left[\frac{1}{n}\sum_{i=1}^{n}|p_i^{\,*}-p_i^{\,j}|\right]$$

Finally, g' is as least as close to h^* as g iff $\Delta_{ms}^{\gamma\gamma'}(h^*, g') \leq \Delta_{ms}^{\gamma\gamma'}(h^*, g)$ and elementary substitutions give us the result of Niiniluoto's truthlikeness definition regarding finite propositional languages.

Without comparing the results of a concrete example, the outcome of the preceding calculation remains opaque. I elaborated the formula for $\mathscr{L}[p,q]$, I calculated the distances between all propositions and the truth $p \wedge q$, and put the results in Table 1. Obviously, these distances depend on the values of the parameters γ and γ', and this raised the question of how the table must be organized. It proved to be convenient to put the true propositions at the left-hand side, and the false at the right-hand side. The distances of the former only depend on γ', as was to be expected, and those of the latter on γ and γ'. The rows of the table only indirectly suggest the distance to the truth $p \wedge q$. If a proposition ψ appears above proposition φ in the table, then ψ is at least as similar to $p \wedge q$ as φ.

Table 1. The min-sum distances of $\mathscr{L}[p, q]$.

$\Delta(p \wedge q, g)$	True g	False g	$\Delta(p \wedge q, g)$
0	$p \wedge q$		
$¼\gamma'$	p, q		
$½\gamma'$	$p \vee q, p \leftrightarrow q$	$\neg p \wedge q, p \wedge \neg q$	$½\gamma+¼\gamma'$
$¾\gamma'$	$p \rightarrow q, q \rightarrow p$	$\neg p \leftrightarrow q$	$½\gamma+½\gamma'$
		$\neg p, \neg q$	$½\gamma+¾\gamma'$
γ'	$\top$	$\neg p \wedge \neg q$	$\gamma+½\gamma'$
		$\neg p \vee \neg q$	$½\gamma+\gamma'$

Table 1 gives rise to two observations. If $\gamma < \gamma'$, $\neg p \wedge \neg q$ is closer to the truth than $\neg p \vee \neg q$. This is a characteristic of *content* definitions similar to those of Popper, Miller and Kuipers. If, $\gamma > \gamma'$, then, in this two-propositional case, the min-sum definition yields that $\neg p \vee \neg q$ is closer to the truth than $\neg p \wedge \neg q$. This agrees with the *likeness* definitions of Hilpinen, and, as we shall see, of Tichý and Oddie, and Brink and Heidema. Thus, in specific cases, γ and γ' determine the question whether the definition yield a content or likeness ordering. For example, if $3\gamma \leq \gamma'$, $\psi <_\tau^\Delta \varphi$ implies $\psi \leq_t^N \varphi$. The parameters γ and γ' also determine the truth-value dependency of the definitions. If $\gamma \gg \gamma'$, the true propositions are all closer to the truth than the false. If $\gamma = \frac{1}{2}\gamma'$, $\neg p \wedge \neg q$ is closer to the truth then $\neg p \vee \neg q$, and some false propositions are closer to the truth than some true propositions. The true and false theories are totally mixed up if $\gamma \ll \gamma'$. Then again, $\psi <_\tau^\Delta \varphi$ implies $\psi \leq_t^N \varphi$. The tables, which reveal these facts can be found in the Appendix.

The significant impact of γ and γ' goes beyond the finite propositional language application. We can easily extrapolate the finite propositional case into more sophisticated situations such as Carnap's *Q*-predicates, and Hintikka's monadic constituents. Regarding Carnap's monadic language $\mathscr{L}_N^k$, the *M*-predicates, *Q*-predicates, and the *Q*-normal form of pure predicates have the same mutual relation as atomic propositions, constituents and propositions in the propositional case. Regarding Hintikka's monadic constituents, the *Q*-predicates, constituents, and (consistent) generalisations also entertain this mutual relation. Consequently, for all applications, the parameters used decide whether the definition yields a likeness or a content ordering.

3.2.3. Remarks and Comments

We saw that Niiniluoto's approach is a strong, quantitative version of Hilpinen's possible world approach, and therefore I introduced it as a likeness definition. Niiniluoto's favourite measure, however, contains two parameters γ and γ' that can be adjusted to the particularities of the circumstances. To be precise, Niiniluoto's proposal is rather a truthlikeness *scheme* or a "two dimensional continuum of truthlikeness measures" than a definition. The two-propositional example shows that the parameters may cause the measure to display content behaviour. There is a price to pay for this versatility, since the introduction of the parameters might impinge on the objectivity of the proposal. In other words, how are we to decide which parameters are appropriate in a particular situation? It seems likely that, in an epistemological context, the adversaries for two competing theories will tune the parameters in favour of their own theory.

The min-sum-measures fulfill a list of thirteen adequacy conditions.[23] The following conditions concern our analysis.

M3. Niiniluoto's proposal avoids Popper's flaw;
M4. an antecedent of a true proposition φ is more truthlike than φ
M5. It avoids the child's-play argument

M8. Adding h_i to g improves g iff $\Delta_{*i} < \Delta_{\min}(h^*, g)$
M10. It may avoid truth-value dependence.

There are more relevant properties than the thirteen mentioned by Niiniluoto. For example, all measures studied by Niiniluoto are vulnerable to extensional substitution. As I shall argue in Chapter 5, this is not a drawback. Further, Niiniluoto's measures fail to be *metrics* on the disjunctive closure of the problem space. They are no distance functions as they are not symmetric. Even the symmetric variant of the min-sum measure, $\delta^{\alpha\alpha'}_{\mathrm{nms}}(g_1, g_2)$, is not a metric; as the next example shows, it fails to have the triangle property. The measure reduces to $M^{\gamma\gamma'}_{ms}(h_1, g_2)$ if g_1 is a complete answer, **B** is balanced and $\gamma = \alpha/|I|$ and $\gamma' = \alpha'/2$. It is defined by:[24]

$$(6) \qquad \delta^{\alpha\alpha'}_{\mathrm{nms}}(g_1, g_2) := \frac{\alpha}{|I|}\sum_{i \in I_{g_1}} \Delta_{\min}(h_i, g_2) + \frac{\alpha'}{|I|}\sum_{i \in I_{g_1}} \Delta_{\min}(h_i, g_1)$$

EXAMPLE: Assume that $\alpha = \alpha'$ such that we may forget $\alpha/|I|$. Suppose that $g_1 := \{i \in \mathbb{N} |\ 1 \leq i \leq 10\}$, $g_2 := \{11, 100\}$, $g_3 := \{j \in \mathbb{N} |\ 101 \leq j \leq 110\}$. Then $\delta(g_1, g_2) = (10 + 9 + ... + 1) + 90 + 1 = 146 = \delta(g_2, g_3)$, $\delta(g_1, g_3) = 2(100 + 99 + ... + 91) = 1910$, and therefore $\delta(g_1, g_2) + \delta(g_2, g_3) < \delta(g_1, g_3)$. The measure violates the triangle inequality. For every choice of α and α' such an example can be found.
End Example

Let us consider the language dynamics of Niiniluoto's proposal. According to Niiniluoto, in the paradigmatic circumstances, the truth is complete, but using (6), he also defines the distances between statements and the distance from the indefinite truth.[25] As mentioned before, if **B** is balanced and $\gamma = \alpha/|I|$ and $\gamma' = \alpha'/2$, $\delta^{\alpha\alpha'}_{\mathrm{nms}}(h_1, g_2) = M^{\gamma\gamma'}_{ms}(h_1, g_2)$. Consequently, if the cognitive problem consists of the constituents of a language $\mathcal{L}$, a conservative extension of the descriptive vocabulary of $\mathcal{L}$ does not change the resulting ordering of the theories. To be more precise, let $\mathcal{L}[p_1,..,p_m]$ be some determinate language and let $\mathcal{L}'[p_1,..,p_m, q_1,..,q_n]$ be the indeterminate extension of $\mathcal{L}$. The equivalence of the δ and M measure implies for all $\varphi, \psi \in \mathrm{Prop}(\mathcal{L})$ and their translations $\varphi', \psi' \in \mathrm{Prop}(\mathcal{L}')$

$$\Delta^{\gamma\gamma'}_{ms}(\tau, \psi) < \Delta^{\gamma\gamma'}_{ms}(\tau, \varphi) \Rightarrow \delta^{\alpha\alpha'}_{\mathrm{nms}}(\tau', \psi') < \delta^{\alpha\alpha'}_{\mathrm{nms}}(\tau', \varphi')$$

Since Niiniluoto distinguishes between the general idea of the cognitive problem and the specific applications of the definitions, the language dynamic behaviour of his proposal regarding the logical vocabulary is a point of controversy. Niiniluoto's truthlikeness definition (scheme) is a paradigmatic example of a definition that is logically biased. In his own terminology, Niiniluoto maintains that his definition depends on the *logical depth* of the analysis. The following example shows what is at issue here.

EXAMPLE: Consider a box partitioned in four equally sized compartments Q_1 ... Q_4, and two indistinguishable balls. These four compartments have the following names: $F(x) \wedge G(x)$, $F(x) \wedge \neg G(x)$, $\neg F(x) \wedge G(x)$, and $\neg F(x) \wedge \neg G(x)$. Logically, there are at least three different syntactic systems for which this box provides models. We can distinguish: 1. the framework of *state descriptions* of monadic predicate language $\mathcal{L}_2^2$ with atomic predicates $F(x)$ and $G(x)$; 2. the system of *constituents* of $\mathcal{L}_2^2$; 3. the system of a *propositional language* $\mathcal{L}[f_1, f_2, g_1, g_2]$. Let the balls be in compartment $F(x) \wedge G(x)$, which is the true theory τ. Let us compare the 'theory' φ claiming that the two balls are in $\neg F(x) \wedge \neg G(x)$, and the theory ψ claiming that one ball is in $\neg F(x) \wedge G(x)$, and the other in $\neg F(x) \wedge \neg G(x)$. Then, the choice of the formal system establishes the ordering of the theories. For those who will object that the balls are indistinguishable, notice that we can call one ball a_1 and the other a_2 if we whish to do so. The table reveals that the ordering of the theories depends on whether they are formulated as state descriptions, monadic constituents, or as propositional constituents.[26] Since the theories are all complete, we do not need the reduction function, thanks to the adequacy condition red($\langle\Delta_{ij}\rangle$) := Δ_{ij}. Niiniluoto chooses the Euclidian distance function between state descriptions. It is the (square root of the normalized) sum over all individuals a_j of the distances between the real Q_i-predicates and the alleged Q_i-predicates.[27] Since ψ's claim about a_1 improves that of φ ($\neg F(x) \wedge G(x)$ is closer to $F(x) \wedge G(x)$ than $\neg F(x) \wedge \neg G(x)$), and they agree about a_2, according to the state description ordering, ψ is closer to τ than φ. Regarding constituents of monadic predicates, however, Niiniluoto chooses the Clifford measure. Since the symmetric difference of the sets of Q_i-predicates of ψ and τ includes the symmetric difference of the Q_i-predicates of φ and τ, φ is closer to the truth τ than ψ. *End Example*

Table 2. Q_i-predicate, and Consistuents

	State Descriptions	*Hintik. Constituents*	*Prop. Constituents*
τ	$Q_1(a_1) \wedge Q_1(a_2)$	$\exists x Q_1(x) \wedge \forall x Q_1(x)$	$f_1 \wedge g_1 \wedge f_2 \wedge g_2$
ψ	$Q_3(a_1) \wedge Q_4(a_2)$	$\exists x(Q_3) \wedge \exists x(Q_4) \wedge \forall x(Q_3(x) \vee Q_4(x))$	$\neg f_1 \wedge g_1 \wedge \neg f_2 \wedge \neg g_2$
φ	$Q_4(a_1) \wedge Q_4(a_2)$	$\exists x Q_4(x) \wedge \forall x Q_4(x)$	$\neg f_1 \wedge \neg g_1 \wedge \neg f_2 \wedge \neg g_2$
ordering	$\psi <_\tau \varphi$ (8.2)	$\varphi <_\tau \psi$ (9.1)	$\psi <_\tau \varphi$

Niiniluoto admits that his proposal depends on the logical machinery of a language. He points out that the distance between the *target* h^* and the statement g determines the truthlikeness of g. The target of truthlikeness is "*the most informative true description of the world* (his italics)", and therefore it depends on the logical *depth* and the descriptive *vocabulary* of the language $\mathcal{L}$. For instance, regarding a monadic language $\mathcal{L}$, state descriptions reveal more structural information than monadic constituents, which have more logical depth than propositional constituents. According to Niiniluoto, there are no a priori arguments to prefer one level to another, and the choice of the appropriate level of logical strength depends on pragmatical considerations. As the logical depth might vary, his approach yields "different concepts of truthlikeness", and generates a "multiplicity of notions of truthlikeness—although they are all treated by the same method."[28]

It is difficult to accept, however, that the comparison of the claims regarding the box with compartments and the two balls depends on the logical representation. Let there be only one ball a, in the in the situation of the preceding example. Then, it is Niiniluoto's use of the Clifford measure rather than the depth of the logical analysis that causes the discrepancy between the state descriptions approach and that of constituents. Let the truth be

$p \wedge q$	which corresponds with
$\exists x Q_1(x) \wedge \forall x Q_1(x)$	(in which $p := F(a)$, and $q := G(a)$)

Then, it is the choice of the logical means that determines the worst theory. In accordance with most other truthlikeness definitions, for appropriate γ and γ' the worst theory of Prop($\mathcal{L}[p,q]$) is

$\neg p \wedge \neg q$	which is true in this model iff
$\exists x Q_4(x) \wedge \forall x Q_4(x)$,	$(\neg F(a_1) \wedge \neg G(a_1))$

According to the Clifford measure, however, the worst theory in $\mathcal{L}^1_2$ is

$\neg p \vee \neg q$ corresponding to

$$[\exists x Q_2(x) \wedge \forall x Q_2(x)] \vee [\exists x Q_3(x) \wedge \forall x Q_3(x)] \vee [\exists x Q_4(x) \wedge \forall x Q_4(x)]$$

If we leave out the propositions of $\mathcal{L}^1_2$ that do not make sense in the model, such as those constituents claiming that there are more than one object in the domain, then the languages distinguish exactly four meaningful constituents. They yield the same meaningful propositions, and it is the Clifford measure rather than the depth of logical analyses that causes the difference in the resulting orderings.

As is clear from the preceding examples, the normalized symmetric difference or Clifford measure has a content character.[29] According to some researchers, this is the reason it cannot be used to measure the distance between monadic constituents. This measure induces a discrepancy between the comparison of finite state descriptions, and the case in which there are monadic constituents with infinite individuals. The state description ordering differs significantly from that of the monadic one; the second is not the limiting case of the first, no matter the seize of the domain of individuals. In sum, for regular γ and γ', Niiniluoto's overall *likeness definition* is based on a *content measure* for distances between constituents.

We will see that Tichý and Oddie in addition to their use of a likeness measure to compare *sets* of constituents, apply a *likeness* measure to fix distances between *individual* constituents.

3.3. TICHÝ AND ODDIE

In 1974 Pavel Tichý was the first (with David Miller) to publish on the flaw in Popper's original verisimilitude definition. In the same publication, he proposed measuring the distance between two constituents using the *mean character difference* (p. 86) (although he did not use this name). In the same issue of the British Journal, David Miller objected to this proposal, since, according to Miller, it yields a 'language dependent' definition. This claim is the subject of Chapter 5 of the present book. Later, Tichý started to cooperate with Graham Oddie, who supported an adapted version of Tichý's original proposal. In 1986, Oddie published *Likeness to Truth*, the second comprehensive book on truthlikeness, and in it he elaborates Tichý's proposals and discusses the differences with Niiniluoto's approach. Oddie maintains that an adequate truthlikeness definition must be able to handle various applications. First it must be able to cope with quantitative applications like "there are exactly twenty planets" is closer to the truth than "there are more than thirty planets." Secondly, it must order propositional applications such as "$\neg h \wedge r \wedge w$" is closer to "$h \wedge r \wedge w$" than "$\neg h \wedge \neg r \wedge \neg w$"; Third, it must handle monadic constituents and propositions, such as the proverbially *fatness, hairiness and tallness* example. Finally, it must order first order structures in which even the importance of the different properties and relations might be weighted. Oddie (1986) is an elaborate treatise on truthlikeness. My introduction will follow (parts of) Oddie (1987a), summarizing the Tichý-Oddie proposal.

3.3.1. The Propositional Definition

Throughout their publications, Tichý and Oddie base their proposals on the *mean character difference*. Regarding two propositional constituents, this measure counts the differences between the literals of the constituents. This only fixes the likeness between (propositional) constituents, and does not provide a likeness definition for weaker propositions of the language. This task requires an answer to the question how to compare sets of constituents with different cardinalities. As we saw, Niiniluoto introduced his reduction function red($\langle \Delta_{ij} \mid j \in \mathbf{I}_g \rangle$) to this end. The averaging methods of Tichý and Oddie

"... measure closeness of fit between two sets of constituents by the breadth of minimal, absolutely fair, linkages: the greater the breadth, the further apart are the two sets."[30]

This section is dedicated to Tichý and Oddies explanation of the preceding quotation. To begin with, I shall explain that the "the breadth of the minimal linkage" fixes the distance between propositions that have the same number of constituents. The intuition underlying this idea is manifest in the left part of Table 3; in it proposition pairs of $\mathscr{L}[h, r, w]$ with the same number of constituents are listed and ordered by increasing mutual distance.

Table 3. Tichy and Oddie Distances

φ	ψ	number const	$D(\varphi, \psi)$	Likeness (φ, ψ)
$(h \vee r \vee w)$	$(\neg h \vee r \vee w)$	7	$^1/_{21}$	$^{20}/_{21}$
$(h \vee r \vee w)$	$(\neg h \vee \neg r \vee w)$	7	$^2/_{21}$	$^{19}/_{21}$
$(h \vee r \vee w)$	$(\neg h \vee \neg r \vee \neg w)$	7	$^3/_{21}$	$^{18}/_{21}$
(h)	$(\neg h)$	4	$^1/_3$	$^2/_3$
$(h \wedge r)$	$(\neg h \wedge \neg r)$	2	$^2/_3$	$^1/_3$
$(h \wedge r \wedge w)$	$(\neg h \wedge \neg r \wedge \neg w)$	1	1	0

What does Tichý mean when he says that the distance between φ and ψ is equal to the *linkage* between φ and ψ with the *minimal breadth*? According to Tichý, the linkage between sets of constituents is defined by a *surjection* between those two sets. If φ and ψ have the same number of constituents this surjection is a *bijection*. Next, the *breadth* of the linkage is the sum of the differences between the related constituents normalised by the total of all possible differences of the linkage. The distance $D(\varphi,\psi)$ between φ and ψ, then, is the minimum of the breadths of all possible distances, and their likeness equals $1 - D(\varphi,\psi)$. The right part of Table 3. surveys the normalized distances and likeness of $\mathscr{L}$-propositions pairs. Let us illustrate these concepts by an example.

EXAMPLE: In $\mathscr{L}[p,q,r]$, $D((h \wedge r),(\neg h \wedge \neg r)) = {}^2/_3$, since $(h \wedge r) \equiv \{(h \wedge r \wedge w), (h \wedge r \wedge \neg w)\}$, and $(\neg h \wedge \neg r) \equiv \{(\neg h \wedge \neg r \wedge w), (\neg h \wedge \neg r \wedge \neg w)\}$, and their minimal linkage is $\{\langle(h \wedge r \wedge w), (\neg h \wedge \neg r \wedge w)\rangle, \langle(h \wedge r \wedge \neg w), (\neg h \wedge \neg r \wedge \neg w)\rangle\}$, since the other pairing gives more dissimilarities. The maximal number of dissimilarity is the number of elements in the linkage (2), times the number of elements in one constituent (3) which makes 6. The number of dissimilarities in the minimal linkage is 2 + 2; therefore the distance between $(h \wedge r)$ and $(\neg h \wedge \neg r)$ is $^4/_6$. The previous diagram illustrates the example. In the same vein, $D((h \vee r),(\neg h \vee \neg r)) = {}^4/_{18}$ and therefore $D((h \wedge r),(\neg h \wedge \neg r)) > D((h \vee r),(\neg h \vee \neg r))$. If the disjunctive normal form of φ and ψ have the same number of constituents, the breadth of the minimal linking is the distance between the two propositions. *End Example*

2 differences

$$D((h \wedge r),(\neg h \wedge \neg r)) := (h \wedge r \wedge w)\ (h \wedge r \wedge \neg w);\ (\neg h \wedge \neg r \wedge w)\ (\neg h \wedge \neg r \wedge \neg w)$$

2 differences

Fig. 5. Example of a linkage

The propositions compared by the minimal breadth definition need not be of the same logical strength. In a propositional context, Tichý (1978) proposes the following. Let φ and ψ be disjunctions of constituents, and let $|\mathrm{Cnst}(\varphi)| = a$, $|\mathrm{Cnst}(\psi)| = b$, and let $a > b$. Then, every *surjection* from Cnst(φ) to Cnst(ψ) is a linkage. The breadth of that linkage is equal to the sum of the differences occurring in each pair of constituents; this is normalised by the sum of all possible differences of the linkage. The distance between two sets of constituents is the breadth of their *minimal* linkage. In other words, every constituent of φ is paired of with a constituent of ψ that is most similar to the constituent in question. For example if $\varphi := \{(h \wedge r \wedge w), (h \wedge r \wedge \neg w), (\neg h \wedge r \wedge w), (\neg h \wedge \neg r \wedge \neg w)\}$ and $\psi := \{(h \wedge r \wedge w), (\neg h \wedge \neg r \wedge \neg w)\}$ then their distance, the linkage of minimum breadth is ${}^{1+1}/_{3\times 4}$, and the likeness between φ and ψ is ${}^{5}/_{6}$. The following diagram illustrates this minimal linkage.

φ		ψ
$h \wedge r \wedge w$	—	
$h \wedge r \wedge \neg w$	—	$h \wedge r \wedge w$
$\neg h \wedge r \wedge w$	—	
$\neg h \wedge \neg r \wedge \neg w$	—	$\neg h \wedge \neg r \wedge \neg w$

Fig. 6. Minimal Linkage

The minimal breadth definition yields a truthlikeness ordering in the two propositional case if the true theory is complete. Let $\mathscr{L}$ be a propositional language with $\mathrm{voc}(\mathscr{L}) := \{p, q\}$, and let the true theory be $p \wedge q$.

The ordering of the minimal breadth definition is given in Table 4, which displays the symmetry regarding the form of the propositions and the negation sign; the fourth row is an axis of symmetry in the table. For example, if the negation signs are interchanged, the formula on the third row is equal to that on the fifth row, those of the second to those on row six, and so on. The disjunction and the conjunction do not participate in this scheme of substitution. The ordering is coarsely grained. Half the propositions receive the same distance to the truth. The table also shows that the minimal breadth definition is a likeness definition. If $p \wedge q$ is the truth, then the absolute falsehood, $\neg p \wedge \neg q$, is the worst element of Prop($\mathscr{L}[p,q]$).

Table 4. Tichý and Oddies two-propositional ordering

	φ	$D(p \wedge q, \varphi)$
1	$p \wedge q$	0
2	p, q	$^1/_4$
3	$p \vee q$	$^1/_3$
4	$p \leftrightarrow q, p \rightarrow q, q \rightarrow p, \top,$ $\neg p \wedge q, p \wedge \neg q, \neg p \leftrightarrow q$	$^1/_2$
5	$\neg p \vee \neg q$	$^2/_3$
6	$\neg p, \neg q$	$^3/_4$
7	$\neg p \wedge \neg q$	1

The next step of our summary is to explain the *absolute fairness* of the linkage mentioned in the quotation. Since Tichý's minimal breadth definition does not put constraints on the surjection, it may distribute the constituents of the larger set unequally over those of the smaller one. Oddie calls such a surjection "unfair". In the previous example, one element of Cnst(ψ) is linked to three elements of Cnst(φ), whereas the other element of Cnst(ψ) only has one counterpart. Oddie proposes to adjust the foregoing definition such that it avoids unfairness. He constrained the definition by reducing the set of allowed surjections, and proposed to allow only *fair surjections*. He defines a fair surjection as follows. Let $|\text{Cnst}(\varphi)| = a > |\text{Cnst}(\psi)| = b$, and let $c \leq {}^a/_b \leq c+1$ (c is an integer). Then, a surjection from Cnst(φ) to Cnst(ψ) is fair if each element of Cnst(ψ) occurs only c or $c+1$ times. This surjection is an improvement, but it still allows for some unfairness. Oddie showed that the fair surjection definition does not provide a metric on the power set of all constituents.

To give a truthlikeness definition that renders a metric on the power set of constituents, Oddie introduces the *absolute fair linkage*. Again let $|\text{Cnst}(\varphi)| = a > |\text{Cnst}(\psi)| = b$, and let c be equal to the lowest common multiple (l.c.m.) of a and b. Then, according to Oddie there is an absolutely fair linkage between φ and ψ if all members of φ are ${}^{c}/_{a}$ times linked to members of ψ, and members ψ are linked ${}^{c}/_{b}$ times to members of φ. For example, regarding $\mathscr{L}[p,q]$, $D(\neg p \vee \neg q, \neg p) = {}^{5}/_{12}$. The l.c.m. of $a = 3$ and $b = 2$ is 6, and the linkage connects all three constituents of $\neg p \vee \neg q$ to the two constituents of $\neg p$. The number of possible mismatches is twelve and the number of actual mismatches is five, therefore, the distance is ${}^{5}/_{12}$. Similarly, we obtain the result that $D(\neg p \vee \neg q, p \vee q) = {}^{1}/_{3}$. Now that I have explained all ingredients of the quotation at the start of this subsection, we are able to understand Tichý and Oddies truthlikeness definition regarding propositional languages.[31] Let $\mathscr{L}$ be finite propositional language, and let $D(\varphi,\tau)$ be the breadth of the minimal, absolutely fair, linkage between φ and τ.

DEFINITION 3.4: Suppose φ, ψ and τ are elements of Prop($\mathscr{L}$); then, according to Tichý and Oddie, ψ *is at least as truthlike* as φ iff $1-D(\psi, \tau) \geq 1-D(\varphi, \tau)$
Notation: $\psi \leq_{\tau}^{T\&O} \varphi$

3.3.2. Questions and Remarks

This second subsection consists of two parts. The first concerns more general issues related to the Tichý and Oddie proposal; in the second, I examine the metatheoretical properties of the definition mentioned in Chapter 1.

Tichý was the first victim of Miller's aversion to "language dependent" approach-to-the-truth definitions. He defended his approach by supplying the language used with intensional semantics. According to Tichý and Oddie, a syntactic truthlikeness definition only makes sense if the formal language on which the definition is based has an explicit interpretation. This is a reasonable condition. If the definition hinges on mere symbol manipulation, it would be difficult to explain how it could contribute to the judgement about closeness to the truth of an empirical theory. To substantiate the claim that syntactic definitions can be used to assess the truthlikeness of an empirical theory, Oddie relates his definition to a Wittgensteinian picture theory of meaning.[32] He claims that propositional constituents contribute to the idea that propositions *picture* states of affairs. Consequently, similarity between states of affairs is related to similarity between constituents. He connects ideas of *logical atomism* with the intuition that similarity between two theories concerns *closeness of fit* between possible worlds. Oddies colleagues in the truthlikeness program do not bring in logical atomism to support their definitions. They agree, however, that the approach of similarity between constituents only make sense if equipped with an appropriate interpretation (see subsection 5.2.3, p. 173).

Less commonly shared is Tichý and Oddies emphasis on the intensional character of this interpretation. The drawback of an extensional interpretation of property terms is that the meaning of the term changes with a change of its extension. According to Oddie, there exists a language independent attribute *red* that provides words such as 'red', 'rot', 'rouge', 'rood'... with meaning, and that does not change when the set of red things changes. More generally, Tichý and Oddie define a *conceptual framework* consisting of a *domain* of objects, and of a collection of language independent traits, the *intensional basis*. A complete distribution of these traits is called a *possible world*, and the collection of all possible worlds produced by the framework is a *logical space*. A *proposition, then,* is a set of possible worlds; it is a dichotomy of the logical space. In this frame of thought, the intension al interpretation induces an "assignment of propositions to sentences."[33] Tichý and Oddie use the classical notion of a proposition: a proposition is that what is expressed by a sentence. We shall see that Tichý and Oddies use of intensional frameworks forms an important part of their reply to Miller's accusation of 'language dependency'.

Tichý and Oddie do not recoil from logical complexity. Apart from their intensional frameworks, they do not stick to first-order logic, and apply higher-order frameworks. According to Oddie, the "universalability of causal connections" suggests that "the primary relation (here: causation) holds not between individual events, but rather between the *properties* of the individuals involved."[34] In the same vain, L.J. Cohen has argued that scientists are looking for better *laws* rather than truth, and proposed to investigate *legisimilitude* instead of verisimilitude.[35] According to Tichý and Oddie their verisimilitude definition provides a definition of legisimilitude as far as the compared propositions make *causal* claims. Consequently, as they interpret causal claims to be higher order traits, their definition must reckon with higher order logic. Tichý and Oddie's treatment of "frameworks with higher-order traits in the intensional basis." falls beyond the scope of this book.[36]

Now, we come to the second part of this subsection, where we shall discuss the metatheoretical properties of the definition. Tichý and Oddies propose a *strong, quantitative truthlikeness* definition, which is *truth-value independent* ($(p_1 \wedge p_2 \wedge p_3) \vee (\neg p_1 \wedge \neg p_2 \wedge \neg p_3) <_{\tau}^{T\&O} \neg p_1 \wedge p_2 \wedge p_3$). Moreover, it is not specular. If $\tau := p \wedge q$, then $p <_{\tau}^{T\&O} \neg p \wedge \neg q$, and, as we saw, $D(\neg p \vee \neg q, \neg p) = {}^5/_{12}$, and $D(\neg p \vee \neg q, p \vee q) = {}^1/_3$, therefore $\neg p \nless_{\neg\tau}^{T\&O} p \vee q$.

As Popper already noted, Tichý's definition is *not upward strictly monotone* (p. 38). The disjunction of a false constituent h_i and a true proposition g is closer to the truth than g, if h_i is closer to the truth than the average distance between the g-constituents and the truth. For instance: $(p_1 \wedge p_2 \wedge p_3) \vee (\neg p_1 \wedge \neg p_2 \wedge \neg p_3) \vee (p_1 \wedge p_2 \wedge \neg p_3) <_{\tau}^{T\&O} (p_1 \wedge p_2 \wedge p_3) \vee (\neg p_1 \wedge \neg p_2 \wedge \neg p_3)$. Tichý's proposal violates Niiniluoto's M4 and M8. This lack of strict monotonicity is the reason for Popper, Miller and Niiniluoto to reject Tichý's definition.

As to the language dynamics of the language, Miller showed that Tichý's original proposal is *context independent*.[37] A comparison in a conservative extension of a language does not contradict the ordering in the original language. Moreover, Tichý avoids the discrepancy between a propositional analysis and an analysis using monadic constituents, such as allowed by the 'logical depth' analysis of Niiniluoto. He treats the propositional, monadic, and first order depth d constituents in the same way. At least as far as these applications are concerned, the approach is *logically unbiased*—that is the change of logical machinery does not change the ordering.

Regarding the quantitative behaviour of the definition, Oddie claims the following. "[I]f a domain has a metric on it, satisfying the triangular inequality, then the distance function between sets of objects in that domain, obtained by using minimal breadths of absolutely fair linkages, is also a metric satisfying the triangular inequality."[38] In other words, the Tichý-Oddie approach *projects the metric* from the level of individuals to the level of sets. Finally, the preceding introduction clearly shows that their proposal does *not suppose the completeness of the truth.*

In the first chapter, I promised to avoid as much as possible the discussion about differences of opinions and intuitions. Here, we cannot evade it. The breadth-of-the-linkage approach treats state descriptions and monadic constituents similarly. Recall the $\mathcal{L}_2^2$-example in subsection 3.2.3 where Q_1 ..., Q_4 are $F(x) \wedge G(x)$, $F(x) \wedge \neg G(x)$, $\neg F(x) \wedge G(x)$, and $\neg F(x) \wedge \neg G(x)$, respectively. Now, consider the constituents $C_1 := \exists x(Q_1) \wedge \forall x Q_1(x)$, $C_4 := \exists x(Q_4) \wedge \forall x Q_4(x)$, $C_{15} := \exists x(Q_2) \wedge \exists x(Q_3) \wedge \exists x(Q_4) \wedge \forall x(Q_2(x) \vee Q_3(x) \vee Q_4(x))$. According to the Clifford (Δ) measure (p. 83) $C_4 <_{C1} C_{15}$, whereas the breadth of the linkage measure yields $C_{15} <_{C1} C_4$ (I substitute Q_i for p_i). Niiniluoto formulates the difference of opinion between him and Tichý thus: "general principle underlying Tichý's judgement ... seems to be this: in measuring the distance between monadic constituents, attention should be paid to the ... number of M_i's and non-M_i's, rather than to the cells Q_i's."[39] This is the reason that Tichý's approach avoids the discrepancy between the (overall) propositional analysis and a monadic constituent refinement. The $M_i(a)$'s and $\neg M_i(b)$'s, or in our terms the $F(a)$'s and $G(a)$'s, correspond to the propositions in a propositional approach. The Q_i-constituents correspond indirectly with those propositions. Consequently, my conclusion reads that, contrary to the Niiniluoto's truthlikeness definition, the Tichý-Oddie proposal is an *overall likeness approach* supplemented with *likeness refinements.*

3.4. BRINK, BURGER AND HEIDEMA

In 1987 C. Brink and J. Heidema proposed a truthlikeness definition for propositional languages based on Brink's ideas about *power relations.* Their definition imposed a preordering on the propositions of a propositional language. It is not antisymmetric as $\psi \leq_\tau \varphi$ and $\varphi \leq_\tau \psi$, does not imply $\psi = \varphi$. In 1994, in cooperation with I. Burger, Heidema changed the 1987 definition into a weaker antisymmetric version; the new version imposed a partial ordering on the propositions. This section is based on Brink and Heidema (1987), and Burger and Heidema (1994).

3.4.1. Brink and Heidema

The Brink and Heidema definition is based on the power relation. Intuitively, a power relation R^+ of a (binary) relation R on set A is the result of transposing R to the powerset of A.

DEFINITION 3.5: Suppose A is some set and let R be some relation on A and let X, $Y \in \wp(A)$. Then, the *power relation* R^+ relates X and Y if and only if
1. $(\forall x \in X)(\exists y \in Y)[xRy]$ and
2. $(\forall y \in Y)(\exists x \in X)[xRy]$

Notation: $X \mathrm{R}^+ Y$

The following example illustrates the idea underlying definition 3.5. Let A be the set $\{1,2,3,4,5,6,7,8,9\}$, and let R be the usual $\leq$-relation. Then, $\{1,2,3,4\} := X \leq^+ Y := \{2,3,4\}$ obtains. In other words, regarding $\leq$, 'Y reaches at least as high as X', in accordance with the first clause of the definition. In the same way 'X is below Y' or 'X reaches lower than Y' according to the second clause. Note that all elements of X are comparable with the elements of Y. Generally, R^+ is a preordering since R^+ is reflexive and transitive if R is; however, $X R^+ Y$ and $Y R^+ X$ do not imply $X = Y$ since R^+ is not antisymmetric. For example, $\{3,5\}\ R^+\ \{3,4,5\}$ and $\{3,4,5\}\ R^+\ \{3,5\}$, but $\{3,4,5\} \neq \{3,5\}$. The power relation can also be applied to a set of sets ordered by the inclusion relation $\subseteq$. Let A be a set. Then, the usual inclusion relation $\subseteq$ partially orders $\wp(A)$, and $\subseteq^+$ preorders $\wp(\wp(A))$. Brink and Heidema introduce a new symbol for the power *ordering*.

DEFINITION 3.6: Let $\subseteq$ order the set $\wp(A)$. Then, the *power ordering* of $\subseteq^+$ on $\wp(\wp(A))$ relates subsets X and Y of $\wp(A)$ if and only if
1. $(\forall x \in X)(\exists y \in Y)[x \subseteq y]$ $(X \leftharpoondown Y)$
2. $(\forall y \in Y)(\exists x \in X)[x \subseteq y]$ $(X \leftharpoonup Y)$

Notation: $X \Leftarrow Y$ ($X \Leftrightarrow Y := X \Leftarrow Y$ and $Y \Leftarrow X$)

Brink and Heidema propose a truthlikeness definition that combines the idea of a power ordering and the fact that every proposition in a propositional language corresponds with an element of $\wp(\wp(\mathrm{voc}(\mathscr{L})))$. The intuition underlying the definition is the following. First, the literals of $\mathscr{L}[p_1, \ldots, p_n]$ are renamed such that all atomic propositions are true. Then, $\mathrm{PC}^+(C_i) \subseteq \mathrm{voc}(\mathscr{L})$ is the set of atomic propositions implied by a *constituent* C_i. Second, as all propositions have a disjunctive normal form, every $\mathcal{L}$-proposition corresponds with a set of $PC^+(C_i)$'s. The inclusion relation orders the subsets of voc($\mathcal{L}$) representing the constituents, and the *power ordering* of this inclusion relation orders all *propositions of* $\mathcal{L}$. Brink and Heidema consider the truth to be the singleton containing only the constituent that affirms all atomic propositions. Let $\mathcal{L}$ be a finite propositional language, and let the truth τ of $\mathcal{L}$ be the constituent affirming all atomic propositions of $\mathcal{L}$, and finally, let $C_i \in \psi$ abbreviate $C_i \in \mathrm{Cnst}(\psi)$. Then, the Brink and Heidema truthlikeness definition reads as follows.

DEFINITION 3.7: Let $\psi, \varphi \in \mathrm{Prop}(\mathcal{L})$; ψ is *at least as close to the truth* τ as φ, iff
1. $(\forall C_i \in \varphi)(\exists C_j \in \psi)[PC^+(C_i) \subseteq PC^+(C_j)]$ and
2. $(\forall C_j \in \psi)(\exists C_i \in \varphi)[PC^+(C_i) \subseteq PC^+(C_j)]$

Notation: $\psi \leq^{\leftarrow}_{\tau} \varphi$ (Brink and Heidema's notation: $\varphi \Leftarrow \psi$)

Regarding the two atomic propositions p and q, Brink and Heidema's definition results in the following ordering. Let the set of atomic propositions of $\mathcal{L}$, voc($\mathcal{L}$) be $\{p, q\}$. Then the four subsets of voc($\mathcal{L}$) are: $\varnothing$, $\{p\}$, $\{q\}$, $\{p, q\}$. Those subsets correspond to the constituents $\neg p \wedge \neg q$, $p \wedge \neg q$, $\neg p \wedge q$ and $p \wedge q$, respectively. In the subsequent explanation I shall use the expression "constituent C_j contains C_i" if $PC^+(C_i) \subseteq PC^+(C_j)$. Moreover, a "maximal constituent C_i of proposition φ" is that constituent of φ that is not contained by any other constituent of φ; a "minimal constituent C_j of a proposition" contains no other constituent of the proposition.

According to Brink and Heidema, constituent $p \wedge \neg q$ is more similar to $p \wedge q$ than $\neg p \wedge \neg q$ since $\varnothing \subset \{p\} \subset \{p, q\}$ and $\{p\} \not\subset \varnothing$. Using the power ordering of the inclusion relation, the definition puts the *proposition* p closer to the truth $p \wedge q$ than $\neg p$. The proposition $\neg p$ is logically identical to $((\neg p \wedge \neg q) \vee (\neg p \wedge q))$, and p is identical to $((p \wedge \neg q) \vee (p \wedge q))$. Thus, $(p \wedge q)$, which is an element of p, contains all constituents of $\neg p$, and $\neg p \wedge \neg q$, an element of $\neg p$, is contained in all the constituents of p. Consequently, p is at least as similar to $p \wedge q$ than is $\neg p$, but since the reverse does not hold, p is closer to the truth than $\neg p$. The preordering that the definition puts on Prop($\mathcal{L}[p,q]$) is shown in Figure 7, which is based on Figure 6 on page 540 of Brink and Heidema (1986).

Using their definition, Brink and Heidema cannot compare all pairs of propositions. For example, although $(p \wedge \neg q) \vee (\neg p \wedge q)$ is a logical consequence of $p \wedge \neg q$, they cannot be compared using the Brink and Heidema proposal. The constituent $p \wedge \neg q$ does not contain $\neg p \wedge q$, and $p \wedge \neg q$ of $(p \wedge \neg q) \vee (\neg p \wedge q)$ is not contained in $\neg p \wedge q$. The definition blocks the comparison in two directions. In contrast to Brink and Heidema's likeness definition, using an content definition we can compare any propositions with its consequences and antecedences.

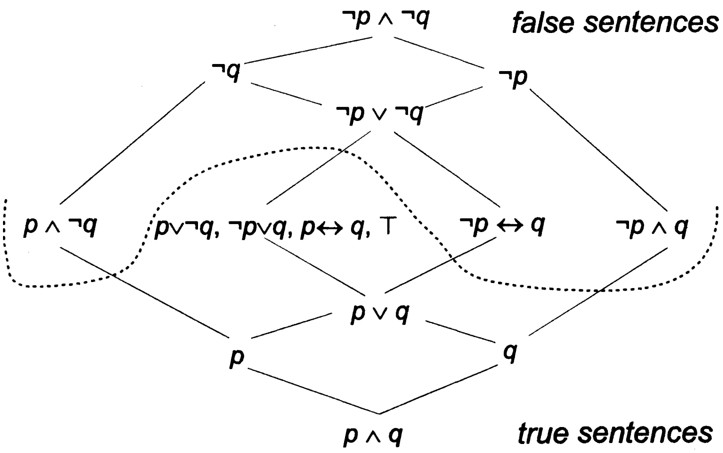


Fig. 7. $\leq_{\tau}^{\leftarrow}$ -ordering of Prop($\mathscr{L}[p,q]$)

3.4.2. Burger and Heidema

In 1994, Heidema and I. C. Burger extended the previous definition from a pre-ordering into a partial ordering of Prop($\mathscr{L}$). For example, according to the power ordering definition, $p \rightarrow q$ is as close to the truth as $p \leftrightarrow q$ since they contain the truth, $p \wedge q$, and the absolute falsehood, $\neg p \wedge \neg q$, and $p \rightarrow q \not\Leftrightarrow p \leftrightarrow q$. Intuitively, Burger and Heidema extend the power ordering definition as follows. If the maximal and minimal constituents of two propositions coincide, then the logical strongest of the two is closer to the truth. If there is no logical consequence relation between the two propositions, their truthlikeness is uncomparable.

First, Burger and Heidema define the *positive* and the *negative* content of a set X of possible worlds (the 1994-paper uses 'possible world', where the 1987-paper used 'constituent'; for finite propositional languages, 'model', 'constituent', and 'possible world' can be mutually substituted). The positive content of X is defined by

$$\vartriangle X :=_{def} \{w \in W \mid \exists x \in X \colon w \geq x\}$$

where $\geq$ designates the *mean character difference* (subsection 3.2.2 p. 86). The positive content of X is the set of possible worlds that improve some model of X. Obviously, it contains the truth. Additionally, Burger and Heidema define the negative content of X:

$$\triangledown X :=_{def} \{w \in W \mid \exists x \in X : w \leq x\}.$$

$\triangledown X$ is the set of structures improved by an arbitrary model of X. Burger and Heidema's next step towards the partial ordering is the definition of the *convex content of* X, and the *non-convex content* of X:

$$\Diamond X :=_{def} \vartriangle X \cap \triangledown X, \text{ and } \text{♂}X :=_{def} (W - \Diamond X) \cup X, \text{ respectively.}$$

The latter equals $(\Diamond X - X)^c$. Intuitively, the convex content of a proposition X is the set of models between the best and the worst models of X. X is convex if $X \equiv \Diamond X$. The non-convex content of X is the complement of the set of worlds that have to be added to X to make it convex. Finally, let $\mathscr{L}$ be a finite propositional language and $W = \mathrm{Cnst}(\mathscr{L})$. then, Burger and Heidema propose:

DEFINITION 3.8: Let $A, B \subseteq W$. Then, B is *at least as close to the truth* as A iff
1. $B \leq_\tau^{\leftleftarrows} A$, and
2. if $A \leq_\tau^{\leftleftarrows} B$ then $\text{♂}A \supseteq \text{♂}B$

Notation: $B \leq_\tau^{\text{♂}} A$

Burger and Heidema show that their definition also can be formulated in terms of the positive and negative content of two theories. They prove that $B \leq_\tau^{\leftleftarrows} A \Leftrightarrow [\triangledown A \subseteq \triangledown B$ and $\vartriangle A \supseteq \vartriangle B]$. The latter is intuitively plausible. After all, $\triangledown A \subseteq \triangledown B$ states that B reaches at least as high as A. In other words, the set of constituents contained by some constituent of A is in the set of constituents also contained by some constituent of B. $\vartriangle A \supseteq \vartriangle B$ says that all constituents containing an element of B are contained in the set of constituents containing an element of A. In other words, A reaches at least as low as B. Furthermore, Burger and Heidema show that $\Diamond A \equiv \Diamond B$ equals $B \leq_\tau^{\leftleftarrows} A$ and $A \leq_\tau^{\leftleftarrows} B$ since then, $\triangledown A \equiv \triangledown B$ and $\vartriangle A \equiv \vartriangle B$. In words this relation between A and B holds if and only if the maximal and minimal constituents of A coincide which the maximal and minimal of B, respectively.

How do Burger and Heidema turn their 1987 preordering into a partial truthlikeness ordering relation? The power ordering definition is similar to the Burger and Heidema definition as long as $\Diamond A \not\equiv \Diamond B$. They diverge when the convex content of theories are the same. Then, the new clause $\text{♂}A \supseteq \text{♂}B$ differentiates between the truthlikeness of A and B. Since $\text{♂}A \supseteq \text{♂}B$ equals $(\Diamond A - A)^c \supseteq (\Diamond B - B)^c$, which equals $\Diamond A - A \subseteq \Diamond B - B$, the extra claim boils down to the following condition: The set of constituents that has to be added to make A convex has to be a subset of the set of constituents that has to be added to B to make B convex. Then, all B-constituents are A-constituents, and A is a logical consequence of B. The result is that if, according to the power ordering definition, two propositions have the same distance to the truth, the logically stronger is closer to the truth. Note that if $\Diamond A \equiv \Diamond B$, $\text{♂}A \supseteq \text{♂}B$, and $\text{♂}B \supseteq \text{♂}A$, then $A \equiv B$, and if $\Diamond A \equiv \Diamond B$, and $\text{♂}A \not\supseteq \text{♂}B$, and $\text{♂}B \not\subseteq \text{♂}A$, the truthlikeness of A and B remain the same.

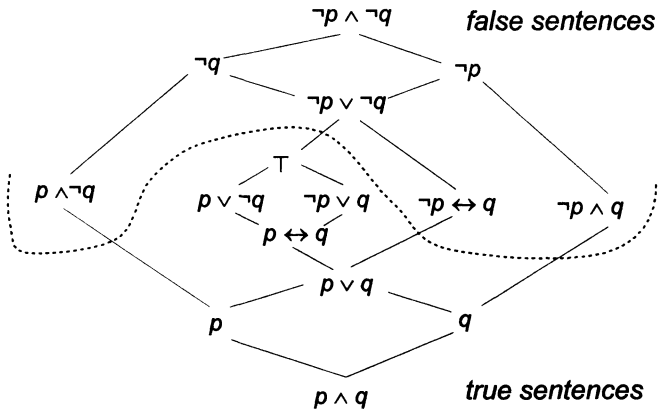


Fig. 8. Burger and Heidema's ordering of Prop($\mathscr{L}[p,q]$)

Finally, it is interesting to see how the Burger Heidema definition copes with the two atomic propositions example. As an illustration, I shall elaborate the comparison of $p \leftrightarrow q \equiv \{\neg p \wedge \neg q, p \wedge q\}$, and $p \rightarrow q \equiv \{\neg p \wedge \neg q, p \wedge \neg q, p \wedge q\}$. The negative content of $p \rightarrow q$ and $p \leftrightarrow q$ are: $\triangledown p \rightarrow q \equiv \triangledown p \leftrightarrow q \equiv \{\neg p \wedge \neg q, p \wedge \neg q, \neg p \wedge q, p \wedge q\} \equiv \top$, and the positive contents of $p \rightarrow q$ and $p \leftrightarrow q$ are: $\vartriangle p \rightarrow q \equiv \vartriangle p \leftrightarrow q \equiv \top$. Consequently, the convex contents of the theories are identical: $\Diamond p \rightarrow q \equiv \Diamond p \leftrightarrow q \equiv \top$, yielding $p \leftrightarrow q \not\Leftrightarrow p \rightarrow q$. The non-convex content of $p \rightarrow q$ equals $p \rightarrow q$, because $♂ p \rightarrow q \equiv (W - \Diamond p \rightarrow q) \cup (p \rightarrow q) \equiv \varnothing \cup (p \rightarrow q) \equiv p \rightarrow q$, and the same holds for $p \leftrightarrow q$: $♂\, p \leftrightarrow q \equiv p \leftrightarrow q$. Consequently, the Burger and Heidema definition applies the convex content clause. This clause yields that $♂ p \rightarrow q \supseteq ♂ p \leftrightarrow q$ since $\{\neg p \wedge \neg q, \neg p \wedge q, p \wedge q\} \supseteq \{\neg p \wedge \neg q, p \wedge q\}$. This shows that where the power ordering definition equalizes the truthlikeness of $p \rightarrow q$ and $p \leftrightarrow q$, the positive and negative content definition favours the logical stronger to the logical weaker one, and: $p \leftrightarrow q \leq^{♂}_{\tau} p \rightarrow q$. The remainder of the two propositional ordering is presented in Figure 8 copied from Burger and Heidema (1994) page 626.

3.4.3. Remarks and Comments

In his review paper about the power ordering definition, Oddie compares the Brink and Heidema proposal with the 'mixed approach'.[40] This is Hilpinen's definition supplemented with the mean character difference measuring the similarity between possible worlds. We used the mixed approach to elaborate Hilpinen's ordering of the Prop($\mathscr{L}[p,q]$) example. Oddie adequately observes that for both approaches the maximal and minimal constituents of propositions determine their truthlikeness. Burger and Heidema define truthlikeness in terms of positive and negative content. If, however, max(X) refers to $\{x \in X | \nexists y \in X: x < y\}$ and min(X) denotes $\{x \in X | \nexists y \in X: y < x\}$, then it is easy to prove that $\vartriangle X = \{w \in W | \exists x \in \min(X): w \geq x\}$ and $\triangledown X := \{w \in W | \exists x \in \max(X): w \leq x\}$.[41] Consequently, Burger and Heidema base their definition on the maximal and minimal constituents of propositions, and it is more similar to Hilpinen's approach than to that of Tichý and Oddie. The latter uses the minimal breadth 'averaging method' instead of the maximal and minimal method. Although Oddie's general assessment is adequate, his claim "that the *power relation proposal and the mixed proposal yield exactly the same judgements* (his italics)" is false.[42] A quick look at Figure 2 (p. 78), Figure 7 (p. 101) and Table 4 (p. 95) suffices to falsify Oddies claim. For example, regarding p and $p \vee q$ and $\tau := p \wedge q$,

$$p \vee q \leq^{H}_{\tau} p, p <^{\leftarrow}_{\tau} p \vee q \text{ and } p <^{T\&O}_{\tau} p \vee q$$

Without plunging into discussions about priorities of intuitions, we may claim that intuitively, regarding the two propositional example, the minimal linkage ordering is at least as similar to the power relation ordering as Hilpinen's ordering.[43] If the vocabulary of the language extends, then the power ordering definition and the minimal linkage definition give different preference orderings, since the second is not truth-value dependent. For example, for $\mathscr{L}[h,r,w]$ with $\tau := \{h \wedge r \wedge w\}$ if

$$\psi := (h \wedge r \wedge w) \vee (h \wedge \neg r \wedge \neg w) \vee (\neg h \wedge \neg r \wedge \neg w), \text{ and}$$
$$\varphi := (h \wedge r \wedge \neg w) \vee (h \wedge \neg r \wedge w) \vee (\neg h \wedge r \wedge w) \vee (\neg h \wedge \neg r \wedge \neg w)$$

$D(\psi, \tau) = {}^{5}/_{9}$ and $D(\varphi, \tau) = {}^{1}/_{2}$, and the false proposition is closer to the truth: $\varphi <^{T\&O}_{\tau} \psi$. As ψ "reaches higher" than φ—ψ is true—and ψ reaches as low as φ, according to the power ordering definition, it is closer to the truth than φ: $\psi <^{\leftarrow}_{\tau} \varphi$.

In the remainder of this subsection, I discuss the other metatheoretical properties of Heidema's definitions. The power ordering and its positive-and-negative-contents weakening are both comparative truthlikeness orderings since nothing reaches lower than, and all other propositions reach at least as high as, the complete falsehood. Both, the power ordering and its weakening are truth-value dependent. Let ψ be a true proposition and φ a false one then, $\varphi \not\leq^{\leftarrow}_{\tau} \psi$. The reason is that if $\tau \vDash \psi$, and $\varphi \vDash \neg\tau$, then the true constituent τ is in Cnst(ψ) and for all C_j in Cnst(φ), $PC^{+}(\tau) \supset PC^{+}(C_j)$. This falsifies the first clause of $\varphi \leq^{\leftarrow}_{\tau} \psi$, and therefore $\varphi \not\leq^{\leftarrow}_{\tau} \psi$; this also proves the truth-value dependency of the Burger and Heidema definition.

As both Heidema's definitions presuppose the completeness of the truth specularity does not apply. It is unclear how they must be adjusted to the case in which the truth is incomplete. The following proposition concerns the language dynamic behaviour of the power ordering.

PROPOSITION 3.5: The power ordering definition is *strongly context independent.*

Proof: By induction on n, the number of atomic propositions. Let $|\text{voc}(\mathcal{L})| = 1$. Then the four propositions are ordered as follows: $\neg p \Leftarrow \top \Leftarrow p$ and $\bot$ is not connected. Figure 7 shows that adding a new atomic proposition q does not change this ordering. Thus, the claim obtains for $n = 1$. If $|\text{voc}(\mathcal{L})| = n$, and $\psi <^{\leftarrow}_{\tau} \varphi$ we shall show that $\sigma(\psi) <^{\leftarrow}_{\tau'} \sigma(\varphi)$ obtains for the translations $\sigma(\psi)$ and $\sigma(\varphi)$ in $\mathcal{L}'$ with $|\text{voc}(\mathcal{L}')| = n + 1$. By definition $\psi <^{\leftarrow}_{\tau} \varphi$ means $\forall C_{\varphi}, \exists C_{\psi}[\text{PC}^{+}(C_{\varphi}) \subseteq \text{PC}^{+}(C_{\psi})]$ and $\forall C_{\psi}, \exists C_{\varphi}, [\text{PC}^{+}(C_{\varphi}) \subseteq \text{PC}^{+}(C_{\psi})]$. Let $\text{voc}(\mathcal{L}') = \text{voc}(\mathcal{L}) \cup \{s\}$. Since φ does not claim anything about s, $\varphi \equiv C_1 \vee \ldots \vee C_n$ iff $\sigma(\varphi) \equiv (C_1 \wedge s) \vee (C_1 \wedge \neg s) \vee \ldots \vee (C_n \wedge s) \vee (C_n \wedge \neg s)$. Thus, $\forall C_{\sigma(\varphi)}, \exists C_{\sigma(\psi)}[\text{PC}^{+}(C_{\sigma(\varphi)}) \subseteq \text{PC}^{+}(C_{\sigma(\psi)})]$ and $\exists C_{\sigma(\varphi)}, \forall C_{\sigma(\psi)}[\text{PC}^{+}(C_{\sigma(\varphi)}) \subseteq \text{PC}^{+}(C_{\sigma(\psi)})]$, and $\sigma(\psi) <^{\leftarrow}_{\tau'} \sigma(\varphi)$. ⊠

The Burger and Heidema ordering weakens the power ordering as the additional constraint renders some pairs of propositions uncomparable whereas according to the power ordering their truthlikeness was equal; for instance $p \to q$ and $q \to p$. As logical implications are preserved under conservative extensions of the language, the foregoing proposition also obtains for the Burger and Heidema definition. Finally, we conclude that, despite the title of their paper, the Burger and Heidema approach leads to an *overall likeness* definition with local *content refinements*.

3.4.4. Kuipers's Truthlikeness

Kuipers considers his truthlikeness definitions as refinements of his content proposals. The Δ-definition, introduced in the preceding chapter, has the benefit of agreeing with common sense. We prefer solution *Y* to *X* if it shares *X*'s advantages, and avoids its drawbacks. The problem of the content definition, however, is its weakness; it will be hard to find applications of the content proposal in the history of science Kuipers conceives his naive definition as an elementary starting point for further refinement, and shows under what conditions his refined definition is equivalent to his content definition. In the first subsection, Kuipers's comparative truthlikeness definition is introduced, and in the next, we shall discuss his refined quantitative proposal. The last subsection concerns remarks, problems and conclusions.

3.4.5. The Comparative Version

In 1987 Kuipers published the first version of his refined definition, and, five years later he changed the definition due to criticism from van Benthem.[44] After the alleged quantitative counter examples of Kieseppä, Kuipers changed his definition again. Regarding the comparative applications, however, this corrected version suffers important drawbacks, which we shall encounter shortly. I regard the newest version to be a step backwards; therefore, I shall base my presentation on Kuipers's 1992 paper.

The basic structuralist starting point of Kuipers's refined comparative definition equals that of his naive definition. Recall that M_p represents the class of all logical possibilities produced by the conceptual framework of the physical theory, and that a subset of M_p represents a theory. The assertion that all *empirical* (im)possibilities belong to X (X^c) embodies the empirical claim of the theory X. Then, the naive definition asserts that Y is at least as close to T as X iff $Y \Delta T \subseteq X \Delta T$. According to the Δ-definition, improvement of a theory only occurs if either some impossible structure has been removed or a physically possible has been added. Since this way of improvement in scientific practice will be rare, Kuipers proposes to refine the definition. Scientific progress consists not only of replacing impossible structures by a possible ones; substitution of an inadequate structure by a better one also provides progress.

How does Kuipers formalize this intuition? First, like Hilpinen, he introduces a basic notion of *similarity* between structures. Let the structures x, y and t be elements of M_p; then, $s(x,y,t)$ says that *y is at least as similar to t as x*. According to Kuipers, this notion of structural similarity cannot be defined, since different types of structures need different definitions of structural similarity. If $s(x,y,t)$ obtains, Kuipers calls y an *intermediate* of x and t; in other words, y lies between x and t. Secondly, the notion of *relatedness* is introduced. Two structures x and z are related, connected, or *comparable*, $r(x, z)$, if there is a $y \in M_p$ such that $s(x,y,z)$. Thus, the meaning of relatedness and similarity depends on the context of their application. Kuipers assumes that the similarity relation s underlying his refined truthlikeness definition meets the following minimal s-conditions: 1. s is *centered*: for all x, $s(x,x,x)$. 2. s is *centering*: $s(x,y,x)$ implies $x = y$. 3. s is *conditionally left reflexive*: $s(x,y,z)$ implies $s(x,x,y)$, $s(x,x,z)$, and $s(y,y,z)$; and it is *right reflexive*: $s(x,y,z)$ implies $s(x,y,y)$, $s(x,z,z)$, and $s(x,z,z)$. Note that the centeredness of s implies $r(x,x)$.

Furthermore, Kuipers distinguishes between *symmetric* and *antisymmetric* similarity relations. If $s(x,y,z)$ implies $s(z,y,x)$, it is symmetric, and if $s(x,y,z)$ and $s(z,y,x)$ together imply $x = y = z$, it is antisymmetric. Consequently, the comparability relation need not be symmetric. Let $\{\langle x, y, z\rangle\}$ define $s(x,y,z)$ in $\{x, y, z\}$. Then, $r(x, z)$ obtains and $r(z, x)$ does not. Hence, if s is not symmetric, it defines a *directed* r-relation. Furthermore, we must be aware that r need not be *transitive.* For example, consider $x := \langle 0,0\rangle$, $y := \langle 2,0\rangle$, $z := \langle 3,0\rangle$ and $a := \langle 0,1\rangle$ in a two-dimensional Euclidian space where $s(p,q,r)$ means the distance between q and r is smaller than the distance between p and r; then, $r(x,z)$ and $r(z,a)$ do not imply $r(x,a)$. Thus, Kuipers's comparability relation need not be transitive nor symmetric. Among Kuipers's examples of symmetric similarity relations are: propositional structures, first order structures and real number structures. The structural likeness triads, generated by concretization, provide important examples of antisymmetric similarity.[45] *Convexity* is the third notion in Kuipers's approach. Set X is convex regarding s if for all x and $z \in X$, and for all $y \in Mp$, $s(x,y,z)$ implies $y \in X$. Kuipers's general truthlikeness definition does not assume theories to be convex.

In Chapter 2, we found that the structuralist content definition has two clauses. The refined definition also consists of two clauses. The naive instantial clause was equivalent to the condition that $(X \cap T) - Y = \varnothing$. *Strengthening* this condition, the *refined* instantial clause reads:

> (Ri) For all x in X and t in T if $r(x,t)$, then there is a y in Y such that $s(x,y,t)$. *Notation*: $\mathrm{RTL}^{\mathrm{in}}(X, Y, T)$

Intuitively, (Ri) claims that if X contains a relevant failure, then there is a $y \in Y$ that is more similar to the physical possible structure t than x. Note that y need not to be an element of T. The naive explanatory clause claims that $Y - (X \cup T) = \varnothing$. *Weakening* this clause, the *refined* explanatory clause reads:[46]

> (Rii) For all y in $Y - (X \cup T)$ there are x in $X - T$, and t in T such that $s(x,y,t)$. *Notation*: $\mathrm{RTL}^{\mathrm{ex}}(X, Y, T)$

According to (Rii), and in contrast to the Cn-clause, Y may contain physically impossible structures that are not in X. The (Rii)-clause demands that these y's are relevant for some t in T and must improve an x in $X - T$ that is also relevant for that t. In other words (Rii) requires that the explanatory mistakes of Y are not worse than the explanatory mistakes of $X - T$. Finally, let M_p be some universe of models, then

DEFINITION 3.9: Let $X, Y, T \subseteq M_p$; according to Kuipers's *comparative truthlikeness* definition Y is at least as close to T as X, *RTL(X,Y,T)*, iff
1. $\mathrm{RTL}^{\mathrm{in}}(X, Y, T)$ and
2. $\mathrm{RTL}^{\mathrm{ex}}(X, Y, T)$.

Notation: $Y \leq_T^{Kr} X$ $\qquad$ $(Y <_T^{Kr} X := Y \leq_T^{Kr} X$ and $X \nleq_T^{Kr} Y)$

Figure 9 epitomizes the comparative likeness definition where all M_p-elements are related by the directed r-relation and all theories are convex. Then, the horizontal line represents the linear ordering of M_p, and line segments represent the theories.

In these circumstances, the maximal and minimal elements of X and Y determine the $\leq_T^{Kr}$-relation. Note that, in contrast to the naive definition, $Y - (X \cup T) \neq \varnothing$.

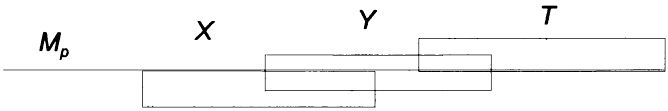


Fig. 9. Kuipers' refined likeness definition

If the comparability relation *partially* orders M_p, and the theories are convex, the picture is more complex (see Figure 10). First we must represent M_p by a Hasse-diagram (Figure 10 presents the Hasse-diagram of Cnst($\mathcal{L}(p,q,r)$)). Then, if X and T are (convex) subsets of M_p, the question reads "what Y improves X?" Intuitively the answer contains three parts. First, for all $\langle x,t \rangle$-pairs in $X \times T$ there must be a path from x to t that contains a $y \in Y$ that "screens x from t"; second, all $y \in Y$ must be "squeezed in" between some $x \in X$ and $t \in T$; and finally Y must be convex—that is, all the intermediates between the "extremes of Y" must be elements of Y. In Figure 10, $Y \leq_T^{Kr} X$ and $Y \not\leq_T^{Kr} X'$ obtain.

I have already referred to the fact that the logical deduction relation connects the naive and refined clauses in different directions. Regarding the *instantial* clauses, the *refined* clause implies the *naive* one. The refined clause is logically stronger than the naive clause since (Ri) implies $(X \cap T) - Y = \varnothing$. As to the *explanatory* side, the deduction relation obtains in the opposite direction. The *naive* clause implies the *refined* one, and it is logically stronger than the refined one; $Y - (X \cup T) = \varnothing$ implies (Rii). Consequently, naive and refined structuralist definitions are not deductively related. $Y \leq_T^{Kr} X$ does not imply $Y \leq_T^{\Delta} X$, nor does the implication obtain in the opposite direction. Refinement of the naive definition is due to the underlying similarity notion, and the worst theory is the complete falsehood. It is a full-blooded truthlikeness definition.

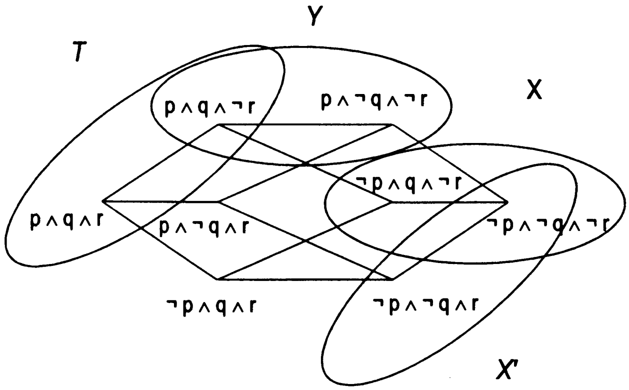


Fig. 10. RTL(X,Y,T) for partially ordered M_p

In the preceding subsection, we saw that Oddie compared the Brink and Heidema approach with Hilpinen's proposal, since the first can be reformulated in terms of the best and worst elements of a theory. The same obtains for Kuipers's refined proposal for convex theories. The set of the best and the set of the worst elements of X may be defined as follows:

$$\max_{s,T}(X) :=_{def} (X \cap T) \cup \{x \in X-T \mid \nexists x' \in X\colon x \neq x' \text{ and } \exists t \in T \text{ with } s(x, x', t)\},$$
$$\min_{s,T}(X) :=_{def} \{x \in X-T \mid \nexists x' \in X\colon x \neq x' \text{ and } \exists t \in T \text{ with } s(x', x, t)\}$$

These sets enable us to give a transparent paraphrase of Kuipers's likeness definition in case the truth is convex. "Y is at least as close to T as X" iff

$$\max_{s,T}(Y) = \max_{s,T}(X \cup Y) \quad \textit{(instantial clause)}$$
$$\min_{s,T}(X) = \min_{s,T}(X \cup Y) \quad \textit{(explanatori clause)}$$

The instantial clause implies $(X \cap T) - Y = \varnothing$, and the explanatory clause is implied by $Y - (X \cup T) = \varnothing$. In other words, for the best elements of X, the theory Y has one element that is at least as good; for all the worst elements of Y, X has an element that is at least as worse. The paraphrase illustrates the intuitions behind Figure 9. It illustrates that Kuipers's likeness definition depends only on the best and worst elements of the theories.[47]

Oddies observation suggests a general comparative truthlikeness definition. Suppose $\preceq_{\mathfrak{W}}$ orders Mod($\mathcal{L}$) towards $\mathfrak{W}$, which represents the truth.

DEFINITION 3.10: ψ is *at least as truthlike* as φ iff
1. $\forall \mathcal{B} \in \text{Mod}(\varphi), \exists \mathcal{A} \in \text{Mod}(\psi)\colon \mathcal{A} \preceq_{\mathfrak{W}} \mathcal{B}$
2. $\forall \mathfrak{M} \in \text{Mod}(\psi), \exists \mathfrak{N} \in \text{Mod}(\varphi)\colon \mathfrak{M} \preceq_{\mathfrak{W}} \mathfrak{N}$

Notation: $\psi \leq_{\mathfrak{W}} \varphi$

The preceding definition may also be formulated in terms of the best models, $\text{Mod}^-(\varphi) :=_{def} \{\mathfrak{M} \in \text{Mod}(\varphi) \mid \mathfrak{M} = \min(\text{Mod}(\varphi) \text{ w.r.t. } \preceq_{\mathfrak{W}}\}$, and and the worst models defined by $\text{Mod}^+(\varphi) :=_{def} \{\mathfrak{M} \in \text{Mod}(\varphi) \mid \mathfrak{M} = \max(\text{Mod}(\varphi) \text{ w.r.t. } \preceq_{\mathfrak{W}}\}$.

PROPOSITION 3.6: $\psi \leq_{\mathfrak{W}} \varphi$ iff 1. $\forall \mathcal{B} \in \text{Mod}^-(\varphi), \exists \mathcal{A} \in \text{Mod}^-(\psi)\colon \mathcal{A} \preceq_{\mathfrak{W}} \mathcal{B}$ and
2. $\forall \mathfrak{M} \in \text{Mod}^+(\psi), \exists \mathfrak{N} \in \text{Mod}^+(\varphi)\colon \mathfrak{M} \preceq_{\mathfrak{W}} \mathfrak{N}$.

Proof: $\Rightarrow$: if $\forall \mathcal{B} \in \text{Mod}(\varphi), \exists \mathcal{A} \in \text{Mod}(\psi)\colon \mathcal{A} \preceq_{\mathfrak{W}} \mathcal{B}$, then $\forall \mathcal{B} \in \text{Mod}^-(\varphi), \exists \mathcal{A} \in \text{Mod}(\psi)\colon \mathcal{A} \preceq_{\mathfrak{W}} \mathcal{B}$. Suppose that $\mathcal{A} \notin \text{Mod}^-(\psi)$, then there is an $\mathcal{A}' \in \text{Mod}^-(\psi)$ such that $\mathcal{A}' \preceq_{\mathfrak{W}} \mathcal{A}$. Hence $\forall \mathcal{B} \in \text{Mod}^-(\varphi), \exists \mathcal{A} \in \text{Mod}(\psi)\colon \mathcal{A} \preceq_{\mathfrak{W}} \mathcal{B}$. $\Leftarrow$: Suppose $\forall \mathcal{B} \in \text{Mod}^-(\varphi), \exists \mathcal{A} \in \text{Mod}^-(\psi)\colon \mathcal{A} \preceq_{\mathfrak{W}} \mathcal{B}$; then a fortiori $\forall \mathcal{B} \in \text{Mod}^-(\varphi), \exists \mathcal{A} \in \text{Mod}(\psi)\colon \mathcal{A} \preceq_{\mathfrak{W}} \mathcal{B}$. Take an arbitrary $\mathcal{B} \in \text{Mod}(\varphi)$; if $\mathcal{B} \in \text{Mod}^-(\varphi)$ then the implication obtains. If $\mathcal{B} \notin \text{Mod}^-(\varphi)$ then the definition of $\text{Mod}^-(\varphi)$ guarantees that there is a $\mathcal{B}' \in \text{Mod}(\varphi)$ such that $\mathcal{B}' \preceq_{\mathfrak{W}} \mathcal{B}$, and the implication holds again. Clause 2. is analogous. ⊠

Furthermore, the general truthlikeness definition provides a preordering. More specifically, the next proposition shows that the set of propositions that are all true on all worst and all best models of the language have the same truthlikeness. More specifically, in a finite propositional language all the elements of Cn($\tau \vee \xi$) are equally truthlike (τ is the complete truth and recall that ξ is the most distant constituent). This is the reason for the Burger and Heidema weakening.

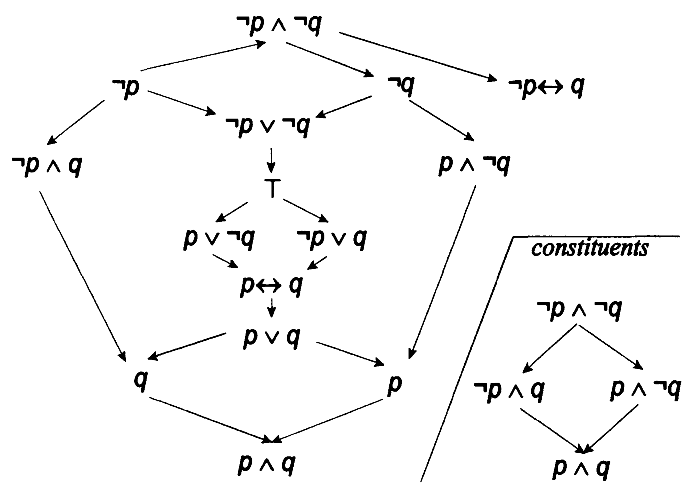


Fig. 11. Kuipers's 1992 ordering of $\mathscr{L}[p,q]$

PROPOSITION 3.7: Let Mod(φ) := Mod$^-$($\mathscr{L}$) $\cup$ Mod$^+$($\mathscr{L}$); then, for all $\varphi' \in$ Cn(φ):
$\varphi' \sim_{\mathfrak{B}} \varphi \sim_{\mathfrak{B}} \top$

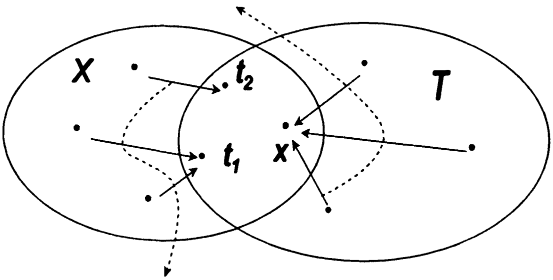
Proof: Let Mod(φ) := Mod$^-$($\mathscr{L}$) $\cup$ Mod$^+$($\mathscr{L}$). If $\varphi' \in$ Cn(φ), then Mod(φ) $\subseteq$ Mod(φ'). Consequently, Mod$^-$(φ') = Mod$^-$(φ) and Mod$^+$(φ') = Mod$^+$(φ), and $\varphi' \sim_{\mathfrak{B}} \varphi$. ⊠

Finally, we turn to the question of how the refined definition fares in the two-atomic propositions example based on the mean character difference. To make the comparison easier I include non-convex theories; otherwise we have to dismiss $p \leftrightarrow q$, $p \leftrightarrow \neg q$, and even $p \rightarrow q$, since they are non-convex. If we neglect the convexity constraint, the Hasse-diagram shows that, except for $p \leftrightarrow \neg q$, the ordering of Kuipers's refined definition is similar to the one of Burger and Heidema (cf. Figure 8, p. 103, and Figure 11).

3.4.6. The Quantitative Version

In Chapter 2, I introduced Kuipers's naive quantitative definition, where the distances between the theories were defined on $\wp(M_p)$. Here, the basic idea is to base the distances between theories on the distances between the elements of the theories. To that end, Kuipers assumes a symmetric semimetric (see p. 31) on M_p, $\mathrm{d}: M_p \times M_p \to \mathbb{R}$, which determines the distances between individual structures of M_p. This distance function replaces the similarity triad of the preceding subsection such that the quantified similarity triad $sd(x, y, t)$ equals $\mathrm{d}(y, t) \leq \mathrm{d}(x, t)$ and $\mathrm{d}(x, y) \leq \mathrm{d}(x, t)$. Kuipers defines the distance between subsets of M_p in two steps.

Distance of the *instantial* problems = $\Sigma_{t\in T}\min\{\mathrm{d}(t, x)|x\in X\}$ = $\mathrm{D}(T/X)$



$\Sigma_{x\in X}\min\{\mathrm{d}(x, t)|\ t \in T\}$ = $\mathrm{D}(X/T)$ = Distance of the *explanatory* problems

Fig. 12. The distance between X and T

First, d: $M_p \times \wp(M_p) \to \mathbb{R}$ fixes the distance between an element of M_p and a subset of M_p. It is the shortest distance between x and Y: $d(x,Y) :=_{def} \min\{\mathrm{d}(x,y)|\ y \in Y\}$. Consequently, the distance from a set to one of its elements is zero: $d(x,Y) = 0$. Second, Kuipers defines the distance D: $\wp(M_p) \times \wp(M_p) \to \mathbb{R}$ *from X to Y*, $D(X\backslash Y)$:[48]

$$D(X\backslash Y) :=_{def} \Sigma_{x\in X}\ d(x,Y)$$

$D(X\backslash Y)$ is asymmetric since, generally, $D(X\backslash Y)$ will differ from $D(Y\backslash X)$. The elements of X that are in the intersection of X and Y do not contribute to $D(X\backslash Y)$ because it is based on $d(x,Y)$, and $d(y,Y) = 0$. Consequently,

$$D(X\backslash Y) = D(X\cup Y\backslash Y) = D(X - Y\backslash Y).$$

Figure 12 shows that the distance between the theory X and the true theory T consists of two parts: the *explanatory* distance $D(X\backslash T)$ and the *instantial* distance $D(T\backslash X)$. The name of the second stems from the fact that $D(T\backslash X) = D(T - X\backslash X)$, and, according to Kuipers, T-X represents the instantial problems of X. It contains empirical possibilities that X excludes. $D(X\backslash T)$ derives its name from the fact that $D(X\backslash T) = D(X-T\backslash T)$, and $X-T$ represents the explanatory problems of X. The set $X-T$ comprises the logical possibilities that X instantiates, but that are empirically impossible. Using $D(X\backslash Y)$, Kuipers defines the distance function $D(X,Y)$: $\wp(M_p) \times \wp(M_p) \rightarrow \mathbb{R}$:[49]

DEFINITION 3.11: The *distance* between X, $Y \subseteq M_p$ is equal to
$D(Y\backslash X) + D(X\backslash Y)$
Notation: $D(X,Y)$

$D(X,Y)$ is an example of a symmetric semimetric. For all X, $Y \subseteq M_p$, it trivially fulfils: $D(X,Y) = 0 \Leftrightarrow X = Y$ and $D(X,Y) = D(Y,X)$. Nevertheless, $D(X,Y)$ does not guarantee the triangle inequality. Let $M_p = \{1, 2, \ldots 20\}$, let $X = \{1, 2\}$, $Y = \{2, 11\}$ and $Z = \{11, 12\}$. Then $D(X,Y) = D(Y,Z) = 10$ and $D(X, Z) = (10 + 9) + (9 + 10) = 38$, and $D(X,Y) + D(Y,Z) < D(X,Z)$; however $D(X,Y)$ suggests a quantitative truthlikeness definition:[50]

DEFINITION 3.12: Suppose X, Y, $T \subseteq M_p$ and let d: $M_p \times M_p \rightarrow \mathbb{R}$ be a symmetric semimetric on M_p. Then *Y is quantitatively at least as close to the truth as X* iff
$D(Y,T) \leq D(X,T)$
Notation: $Y \leq_T^D X$

Under special conditions, Kuiper's naive quantitative definition equals his refined definition. The trivial metric, d^n: $M_p \times M_p \rightarrow \mathbb{R}$ is the *naive distance* on M_p iff: $d^n(x, y) = 0$ if $x = y$, and $d^n(x, y) = 1$ if $x \neq y$.[51] Kuipers's proves that, $Y \leq_T^D X$ equals $Y \leq_T^{Dn} X$ iff the first is based on d^n.

The comparative refined proposal is not a limiting case of the refined quantitative one. For false theories, an increase of logical strength may cause an increase of the refined comparative truthlikeness, but a decrease of refined quantitative truthlikeness. This makes Kuipers sceptical about the use value of the refined quantitative approach. Since his comparative definition fails to imply the quantitative version in all situations, the second will not account for idealization concretization examples ordered by the comparative definition.

3.4.7. *Remarks and Comments*

This subsection consists of four parts. In the first one we consider the metatheoretical properties of Kuipers's comparative and quantitative proposals, and in the second, I present Kuipers's Reduction Theorem. In the third subsection, we will examine the latest version of the comparative refined definition; and in the fourth subsection I shall explain why the comparative likeness definition does not relate to the quantitative one.

3.4.7.1 The metatheoretical properties

The structuralist comparative refined proposal is a paradigmatic example of a truthlikeness definition. All propositions of a propositional language, the negation of the truth included, are better than the complete falsehood ξ, and the ordering is likely to change under extensional substitutions, the alleged 'language dependency' property. The *quantitative* version, however, has a *content* character, since there is no theory worse than the negation of the truth. Moreover, according to the quantitative refined version, the weaker theory is less similar to the truth, i.e. it is subject to the child's-play objection. The reason for this behaviour is obvious. Kuipers's measure adds all the differences between a theory and the truth, and does not take the *average* of these distances. In the next chapter, we shall see that the refined quantitative definition transforms into a likeness definition if its measure is based on the average distance between the elements of a theory and the truth. Niiniluoto (1986a) also proposes this content definition.[52] The content and likeness character of the various structuralist proposals are displayed in Table 5.

Kuipers's quantitative truthlikeness definition is stronger than his comparative one. Avoiding overall restriction, the first relates more theories than the second. The comparative one requires that for all the best elements x of the worst theory, there must be elements of the better theory that improve, or are equally good as, x. Moreover, all the worst elements y of the best theory must be worsened by, or be equally good as,elements of the worst theory, and because of a different averaging method, the quantitative version avoids such absolute restrictions.

Table 5. Kuipers's Taxonomy

Definition	*Comparative*	*Quantitative*
Naive	Content	Content
Refined	Likeness	Content

PROPOSITION 3.8: Kuipers's comparative truthlikeness definition is *truth-value dependent*.

Proof: We must show that if φ is true and ψ is false, then $\psi \nleq_T^{Kr} \varphi$. Let φ be true, that is: $\mathrm{Mod}(\tau) \subseteq \mathrm{Mod}(\varphi)$, and let ψ be false such that $\exists t \in \mathrm{Mod}(\tau)$: $t \notin \mathrm{Mod}(\psi)$. Regarding that t, $\nexists b \in \mathrm{Mod}(\psi)$: $s(t,b,t)$. Therefore (Ri) is falsified and $\psi \nleq_\tau^{Kr} \varphi$. ⊠

PROPOSITION 3.9: Neither the comparative (1) nor the quantitative (2) structuralist truthlikeness definition is specular.

Proof: Two counter examples do the job. (1). Let $M_p := \{x, y, z\}$ such that $s(x,y,z)$. If $X := \{x\}$, $Y := \{x, y\}$, and $T := \{z\}$, then $Y <^{Kr}_T X$. The negations of the theories are the complements in M_p and therefore $\neg Y \nless^{Kr}_{\neg T} \neg X$. (2). Suppose $M_p := \{1, \ldots, 10\}, X := \{10\}, Y := \{6, \ldots, 9\}, T := \{1, \ldots, 5\}$. Then $D(X,T) = 40 > D(Y,T) = 25$; however, regarding their complements $D(\neg X,\neg T) = 16 < D(\neg Y,\neg T) = 21$ obtains. ⊠

PROPOSITION 3.10: In propositional context, Kuipers's comparative truthlikeness definition is *strongly context independent.*

Proof: The proof by induction is analogous to the proof of proposition 3.5 (p.105), and is left to the reader. ⊠

Finally, Kuipers's assumes the truth to be incomplete. In the preceding chapter, however, I showed that the structuralist intuitions about this incompleteness must be formalized using a modal language. The modal formulations may incorporate the likeness considerations of the current section.

3.4.7.2 Kuipers's Reduction Theorem

Kuipers proved a *reduction theorem* linking his naive and refined definition. Let $t(x, y, z)$ be the trivial similarity relation defined by $t(x, y, z) :=_{def} x = y = z$, and let $Y \leq^{Kt}_T X$ mean that $Y \leq^{Kr}_T X$ obtains on the basis of the trivial similarity relation.

PROPOSITION 3.11: For all $X, Y, T \subset M_p$: $Y <^{\Delta}_T X$ iff $Y <^{Kt}_T X$

Proof: "$\Rightarrow$" *Instantial*: Let $x \in X$, $z \in T$; then, if $r(x, z)$ then $x = z \in X \cap T$, and $x \in Y$ and for this $y = x \in Y$ holds $t(x, y, z)$. *Explanatory*: Generally, for all $s(x,y,z)$, (Nii) implies (Rii), $t(x, y, z)$ is no exception. "$\Leftarrow$" *Instantial*: Generally, for all $s(x,y,z)$, (Ri) implies $(X \cap Y) - T = \varnothing$, and $t(x, y, z)$ is no exception. *Explanatory*: For all y in $Y - (X \cup T)$ there are x in $X - T$, and z in T such that $t(x, y, z)$ therefore $y \in X \cap Y \cap T$, and $Y - (X \cup T) = \varnothing$. ⊠

The most important difference between the naive and refined definition is that the second does not exclude $Y - (X \cup T) \neq \varnothing$. Intuitively, proposition 3.11 shows that if the similarity relation is the identity relation, the elements of $Y - (X \cup T)$ must be "in between elements of $X - T$ and elements of T; consequently $Y - (X \cup T)$ must be empty.

Although the preceding proposition is straightforward, it is open to an interpretative mistake. To avoid this error one must read the proposition "from the naive side to the refined side." In other words, one can interpret a naive theory comparison as a refined comparison based on the trivial similarity relation. Reading the proposition in the other direction, viz. "from the refined to naive," makes no sense as substituting s by the trivial ordering t considerably changes the theories and the cognitive problem. Consider domain $M_p := \{x, y, z\}$ ordered alphabetically by s. The truth T, and theories Y and X are defined by $\{z\}$, $\{x, y\}$, and $\{x\}$, respectively. According to the refined definition Y is better than X: $Y <_T^{Kr} X$ and not $X <_T^{Kr} Y$. The naive definition obtains in opposite direction: $X <_T^{\Delta} Y$ and not $Y <_T^{\Delta} X$, and the naive ordering contradicts the refined ordering. Changing the alphabetic order s into the trivial order t, however, implies $x = y = z$, and $X = Y = T$; consequently, the naive and refined ordering are the same. The example shows that the proposition must be read from naive to refined: $X <_T^{\Delta} Y$ equals $X <_T^{Kt} Y$ if we impose the trivial similarity relation on M_p such that $s(x,y,z)$ means $x = y = z$. Then indeed, $X <_T^{Kt} Y$ and not $Y <_T^{Kt} X$. Let us call $s(x,y,t)$ empty in M_p if there is no $\langle x,y,t\rangle \subseteq M_p$ such that $x \neq y \neq t \neq x$ and $s(x,y,t)$; then, proposition 3.11 gives rise to the following generalisation:

PROPOSITION 3.12: For all X, Y, T in M_p, $Y <_T^{\Delta} X$ equals $Y <_T^{Kr} X$ iff $s(x,y,z)$ is empty in M_p.

Proof: $\Rightarrow$: Let a non-empty $s(x,y,z)$ obtain in M_p. Then the X, Y, T from the preceding example show that the $<_T^{Kn}$- and $<_T^{Kr}$-ordering are not identical. $\Leftarrow$: see the proof of proposition 3.11. ⊠

Whether the term 'reduction theorem' for proposition 3.11 is a misnomer is a matter of taste. The lemma says that $\leq_T^{Kr}$ is equivalent to $\leq^{\Delta}$, if the similarity s relation for $\leq_T^{Kr}$ is the empty relation on M_p. In our example, the $\leq_T^{Kr}$-ordering has virtually nothing to do with the $\leq_T^{Kt}$-ordering; and the $\leq^{\Delta}$-ordering is certainly not a "borderline case" of the $\leq_T^{Kr}$-ordering such that $Y \leq_T^{\Delta} X \Rightarrow Y \leq_T^{Kr} X$, and $Y \leq_T^{Kr} X \Rightarrow (Y \leq^{\Delta} X$ or $Y \gneqq^{\Delta} X)$. In this sense, for all X, Y the Δ-definition implies The consequence definition, and the preordering of Brink and Heidema implies the one of Burger and Heidema.

The conclusion of this subsection reads that Kuipers's "reduction theorem" does not qualify the fundamental difference between likeness and content approaches. The $\leq^{\Delta}$- and $\leq_T^{Kt}$-relation are identical, and differ from $\leq_T^{Kr}$-relation. The naive ordering is preserved under extensional substitutions, it is open to the child's-play objection, and it considers the negation of the truth to be the worst possible proposition. The $\leq_T^{Kr}$-ordering avoids the child's-play objection, it renders the absolute falsehood as the worst proposition, and it may alter under extensional substitutions (Section 5.2.4, p. 175).

3.4.7.3 Convexity and the Flaw of the 1997-Version.

The convexity constraint plays an important role in Kuipers's likeness approach; and it provoked some hard criticism. Kuipers admits that "... the restriction to convex theories is certainly not unproblematic."[53] Kieseppä (1995) criticises the convexity assumption. Although his motivation is correct—drop the convexity assumption—his examples are, to say the least, far-fetched. Very generally, the examples concern competing theories, or rather guesses, about the amount of energy of a specific vibrating harmonic oscillator. Although, the oscillator bounces smoothly through space, Kieseppä shows that if we plot in one diagram its velocity and the position as function of time, we get an ellipse. Consequently, given a suitable similarity function s, estimations about the kinetic *or* potential energy in the oscillator produce non-convex graphs; and Kuipers's comparative likeness definition allegedly produces counter intuitive results.

Although the oscillator example does not conceal formal errors, one may seriously doubt whether it respects the principle of charity. The phenomenon can hardly be held against the structuralist. He cannot prohibit that an arbitrary combination of system variables, combined with a suitable similarity function, results in a non-convex graph. The velocity of a harmonic oscillator is not a function of its position, nor does the reverse of this statement hold. Both are functions of time and are only *indirectly* related. After all, there is no jump or non-convexity in the empirical possibilities of the positions and velocities of a harmonic oscillator. The example only shows that not every reformulation of essentially convex, time dependent properties is again convex. Guessing the energy in a classical harmonic oscillator, however, seems to be an example of a convex cognitive problem. A correct example of a non-convex system would be an oscillator with *quantized energy* as in Planck's solution for the ultraviolet catastrophe.

In response to the alleged counter examples of Kieseppä, in 1997 Kuipers altered his refined definition.[54] Confronted with this example, he decided to keep the instantial clause, and change the explanatory clause into:

> (Rii′) For all y in $Y - (X \cup T)$ there are x in $X - T$, and t in $T - X$ such that $s(x,y,t)$.

Thus, whereas according to (Rii), t has to be an element of T, (Rii′) wants it to be a member of $T - X$. This adaptation of the definition has a serious drawback, witness the following example. Suppose x, y and t are members of M_p such that: 1. $x \neq y \neq t \neq x$ and 2. $s(x,y,t)$. For example, $x := \neg p \wedge \neg q, y := p \wedge \neg q$, and $t := p \wedge q$. Furthermore, let the theories be $X := p \leftrightarrow q,\ Y := p$, and $T := p \wedge q$. According to all the likeness orderings in this chapter, Y is at least as close to the truth T as X. Kuipers (1992) truthlikeness definition also produces this outcome. The first clause is fulfilled, since the similarity relation on structures is reflexive, and the elements of M_p are comparable. X, Y, and T also fulfill (Rii) of the original definition, since $s(x,y,t)$ obtains for $t \in T$. (Rii′), however, does not obtain since $T - X = \varnothing$, and there is no t such that $s(x,y,t)$. As the second clause is not fulfilled, according to the new definition, Y is not closer to T than X. Kuipers, however, retains his latest version, arguing that although it fails to produce the right answer it does not produce a wrong one.

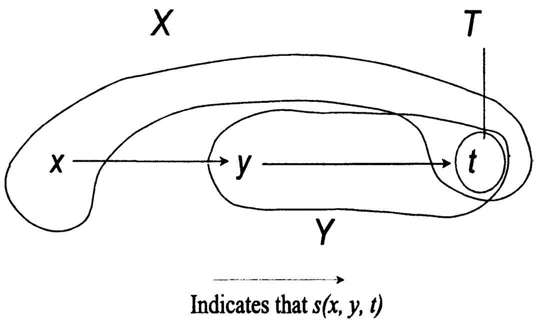


Fig. 13. Counterexample RTL-1993

More generally, the modified definition orders true X and Y only if Y-$(X \cup T) = \varnothing$, or if X is a logical consequence of Y. At the outset, the main reason for introducing the refined definition was that replacing false answers of a theory by still false, but better answers must improve the original theory (see also Figure 9, p. 108). The latest version of the structuralist proposal fails to establish this feature for true propositions.

Kieseppä agrees that the convexity of theories is not an absolute but a context dependent affair.[55] Guesses about the amount of energy of a harmonic oscillator are not convex if formulated as functions of velocity and place. They are convex if formulated straightforwardly in terms of energy, and they do not cause troubles for Kuipers's (1992) refined definition. The drawback of Kuipers's 1992-version does not counterbalance the more important flaw of the 1997-version.

3.4.7.4 Problems with Quantitative Proposals

A considerable amount of time and energy has been spent to answer the question

(*) "whether or under what further conditions qualitative truthlikeness is compatible with quantitative truthlikeness"[56]

This problem is not only of systematical interest; it expands the possibility to apply the structuralist truthlikeness definition. Kuipers accepted the challenge of Laudan. In Chapter 5, we shall see that at least in the comparative *naive* case, theories that are closer to the truth are bound to have more empirical success. The important examples of scientific progress, however, concern quantitative theories and evidence, and as long as the distances among individual structures are determined in an ad hoc manner, the procedure leads to rather arbitrary outcomes. A link between the quantitative and qualitative refined definition would contribute to a structuralist proposition about evidence predominance of the theory that is (quantitatively) refined closer to the truth.

Despite all efforts, I did not manage to come up with a satisfactory answer to question (*), and, considering the verisimilitude and truthlikeness distinction, it seems very unlikely that such an answer exists. Moreover, considerations about the "averaging methods" of the quantitative and comparative versions also add to the improbability of a positive answer. In the first place, Table 5 (p. 113) shows an important difference between the comparative and quantitative truthlikeness definitions. The quantitative definition has features of a content proposal, whereas the comparative refined approach has not. This is not only a difference of taxonomy. Regarding false propositions, the first favours weak to strong ones, and the second has the reverse preference ordering. In the second place, the proposals use different averaging methods. The comparative definition is based on the extremes of a theory, and disregards all intermediate elements. The quantitative version uses the sum of the distances between two theories, taking all the intermediates into account. Without drastic changes it seems impossible to relate the comparative proposal to the quantitative one.

The conclusion of this section reads that Kuipers is the only one to propose content *and* likeness definitions. He has explored the possible combination of his formal naive and refined proposals, and shown that for the empty similarity relation his truthlikeness proposal transforms into a content definition. This transformation, however, does not establish a gradual transition from a content to a likeness definition; it establishes an entirely new ordering. As was to be expected, the most obvious difference between the content and likeness ordering concerns the worst proposition (see Figure 11, p. 110 and Figure 4, p. 50). Finally, the attempt to relate the refined structuralist definition to the quantitative refined version failed; and without readjustments, the structuralist likeness and content orderings are incompatible.

3.5. SUMMARY AND PROSPECTS

In this section, I summarize some important findings of the Chapters 1–3. These findings concern my survey of a major part of the approach-to-the-truth investigations initiated by Popper's 1963-verisimilitude definition, and its 1974-refutation of Miller and Tichý.

3.5.1. Summary

1. Popper's ideas about verisimilitude initiated two directions in the approach-to-the-truth research: We met content and likeness proposals, which define different ideas, called verisimilitude and truthlikeness.

2. The following are the most important characteristics of *verisimilitude* definitions.
   a. Neglect of similarity among possible worlds; verisimilitude definitions are based on *logical strength* and *truth-value*.
   b. They consider the *negation of the truth* as the worst possible theory.
   c. The non-modal versions are *vulnerable to the child's-play objection.*
   d. Verisimilitude orderings remain *unchanged under extensional substitutions*, this feature will be explained extensively in Chapter 5
3. The following are the most important characteristics of the *truthlikeness* definitions.
   a. Truthlikeness is primarily based on *similarity between possible worlds* whereas logical strength only plays a subsidiary role.
   b. The *complete falsehood is the worst* possible theory.
   c. Immunity to *the child's-play objection.*
   d. Possible *change of order under extensional substitutions.*
4. In accordance with Oddies suggestion, we came to the conclusion that all *comparative truthlikeness definitions* of the present chapter—those of Hilpinen, Heidema, and Kuipers—can be reformulated in terms of *best and worst elements.* Their proposals provide very similar orderings on the set of propositions of a propositional language.
5. I showed that all published *comparative approach-to-the-truth proposals* considered yield *truth-value dependent orderings*; only the quantitative proposals avoid this feature. These quantitative proposals, however, are based on rather arbitrary averaging methods.
6. All truthlikeness proposals we considered are *two-stage affairs*. First, they define a similarity ordering on Mod($\mathscr{L}$); and subsequently they transfer this similarity ordering to the level of the $\mathscr{L}$-propositions, $\wp$(Mod($\mathscr{L}$)).
7. The two-stage procedure enables authors to *combine likeness and content considerations*.
   a. For regular γ and γ', Niiniluoto's proposal is an example of a likeness definition with a content ordering of constituents, the Clifford measure.
   b. Tichý and Oddie base their overall likeness approach on a likeness ordering of constituents; a feature shared by the Brink-Heidema approach.
   c. The Burger-Heidema refinement adds content considerations to Heidema's original proposal, as the strongest of two theories with the same truthlikeness is the closest to the truth.

Some of our findings are presented schematically in Figure 15. It displays the skeleton of the approach-to-the-truth research initiated by the publication of the flaw in Popper's 1963- definition.

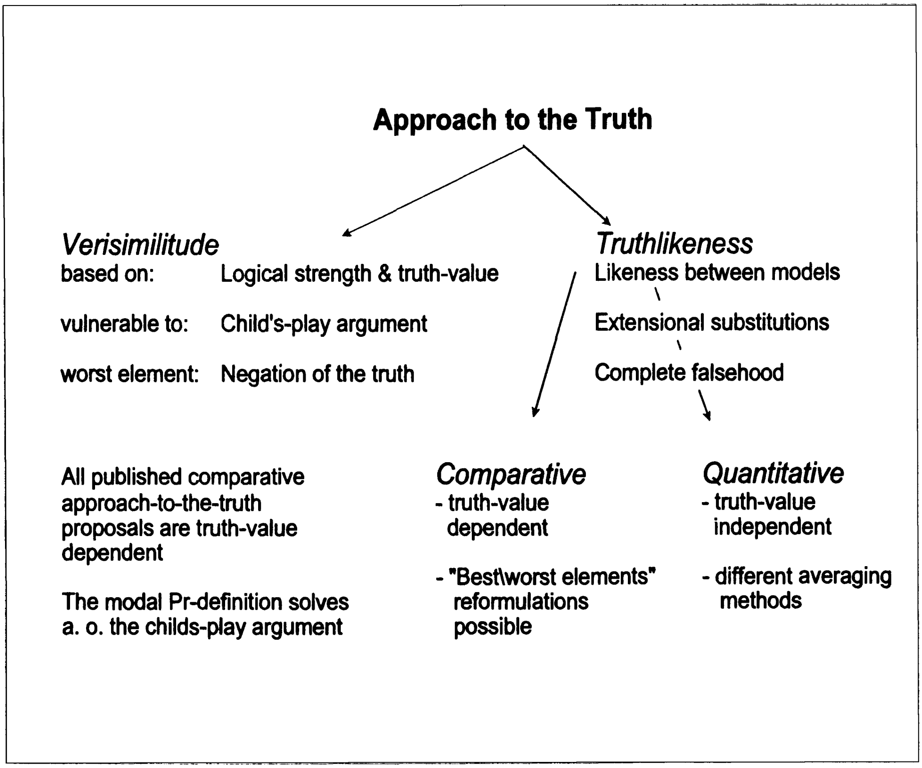


Fig. 14. The approach-to-the-truth research

3.5.2. Prospects

We did not encounter a general *content definition* incorporating *likeness considerations.* As Miller requires invariance under extensial substitutions, he rejects every attempt to take likeness considerations into account. In Chapter 5, however, I shall take the edge off Miller's so-called "language dependency" argument and show that change of ordering after extensional substitution is harmless. Thus, I shall pave the way for my refined verisimilitude definition in Chapter 6 that combines the best of both worlds. On the one hand, its overall content framework systematically uses logical strength and disregard the ordering of models; on the other hand, its likeness refinement systematically use similarity between models and disregards logical strength. Before entering the preparatory Chapter 5 and the grand finale of the sixth chapter, first, I shall address the epistemological question of approach to the truth in Chapter 4.

CHAPTER 4

THE EPISTEMIC PROBLEM

In the first chapter, I mentioned the difference between the *semantic* (1) and the *epistemic* (2) problem of approach to the truth, and I have dedicated the first three chapters of this book to the first problem. In the present chapter, we shall study a more practical subject, viz. the second problem and three answers to it. Let me formulate the two questions explicitly:

(1) *The semantical problem:* "What *do we mean* if we claim that the theory ψ is closer to the truth than φ?"

(2) *The epistemic problem:* "On what evidence *are we to believe* that the theory ψ is closer to the truth than φ?"

In Section 4.1, we encounter general features of the answer to (2), and in Section 4.2, we will discuss the combination of Popper's falsificationism and verisimilitude. Kuipers's and Niiniluoto's answer to the epistemic problem are the subject of the Sections 4.3–4.4; the differences between the approaches of Niiniluoto and Kuipers are dealt with in Section 4.5.

4.1. GENERAL INTRODUCTION

The most important difference between the semantical and the epistemic approach-to-the-truth problem concerns our knowledge about the true theory. Defining the *meaning* of "approach to the truth" is an analytical affair; and we saw in the preceding chapters that an answer to question (1) may presuppose complete knowledge of the truth. Such an answer explains under what circumstances a theory is closer to the truth than another one. When solving the epistemic problem, however, we may not assume acquaintance with the true theory; scientists do not know the truth when carrying out empirical investigations. They must choose between competing theories, on the basis of restricted information about the truth; thus, an answer to (2) is a rule of theory-choice.

After Miller's and Tichý's refutation of Popper's definition, many researchers took up question (1); and their answers fulfilled various conditions of adequacy. One such condition is the possibility to equip the definition with a suitable rule of theory-choice. Among the proposals introduced in the preceding chapters, only

Popper, Niiniluoto and Kuipers answered the epistemic question. Popper (1963, p. 235) was convinced that his "theory of testability and corroboration by empirical tests is the proper methodological counterpart to" the idea of verisimilitude; Niiniluoto (1977) used the estimated degree of truthlikeness, based on probability distributions over all complete answers to a cognitive problem; and Kuipers (1982) proposed a rule inspired by his content definition. Adding convexity constraints, he adapted the rule to his (comparative) likeness approach. Before presenting the strategies of Popper, Kuipers and Niiniluoto, the remainder of the subsection deals with three global evaluation criteria for the rules.

4.1.1. Evidence and Congruence between the Definition and Rule

In the first chapter we encountered the difference between the *mathematical application* of a proposal—such as propositions, numerical vectors and so on—and the *intuitive interpretation* of these applications. Here, we must consider a similar question; it reads:

how does the paraphrase of the theories represent available evidence?

Most authors use the received, or statement view in which evidence is represented by logical consequences. Popper and Miller epitomize this traditional stand. They represent a theory as a deductively closed set of consequences of an axiom φ, $Cn(\varphi)$. If sentence e paraphrases the evidence, and corroborates the theory, then $e \in Cn(\varphi)$. Moreover, as we have seen, Popper and Miller assume that the truth is complete. We saw in Chapter 3 that Niiniluoto's general framework represents propositional theories as disjunctions of constituents. Theories are partial answers to a cognitive problem in which the truth is one complete answer. We shall see that Niiniluoto bases his rule of evidence on the true logical consequences of all confirmed complete answers h_i.[1]

Kuipers deviates from this well-trodden path as far as evidence is concerned. Convinced of the practical advantages of the Suppes-Sneed method, he claims that in a universe of logical possibilities, the true theory picks out all those logical possibilities that are also physically possible, and nothing else. In this framework, one element of the structuralist M_p-set corresponds to one logical possibility. Consequently, if the empirical evidence increases, the set of established, physically realizable structures also increases. This feature of the structuralist framework contrasts with the received view, where increase of the set of models implies decrease of logical strength and, hence, of realizable logical consequences.

Another significant question concerns the correspondence between the "order-fixing elements" of the definition and the rule. Order-fixing elements are the entities that determine the order of the definition or the rule. For instance, the order-fixing elements of the Δ-definition are logical consequences—the more true consequences the better the theory. Thus, we expect that the logical consequences also establish the ordering of the corresponding rule. More generally, we would expect that the elements fixing the ordering when we have complete knowledge of the truth, are the same as those that settle the ordering when we have only partial knowledge of the truth. As it turns out, this does not hold for all answers to the epistemic problem.

4.1.2. Reversible or Irreversible Rule

Most researchers agree that the rule of theory-choice must be *revisable*—that is, in the light of new evidence, the rule must allow for the withdrawal of an old preference ordering. A more difficult question concerns the *reversibility* of the rule. The question reads whether new evidence may cause an adequate rule to reverse its previous strict preference order. Closely connected to the issue of reversibility is the question whether a rule is "*functional for approaching the truth*" (the phrase is Kuipers's).[2]

DEFINITION 4.1: An epistemic rule ρ is (weakly) *functional for approaching the truth* if for all evidence e, ρ's preference of ψ on basis of e, $\psi <_e^\rho \varphi$, implies $\varphi \nless_\tau \psi$ and leaves open the possibility that $\psi <_\tau \varphi$.[3]

The reversibility question has been answered from two different angles. The first one calls on an adequacy argument, and the second uses theoretical considerations. The first argument says that in scientific practice, functional or even irreversible epistemic rules do not exist. On the contrary, scientists make progress only by conjecture and refutation; and no method evades reversibility. Niiniluoto claims "it is one of my adequacy conditions —rather than a 'major difficulty'—that the appraisals by ver are fallible and revisable".[4] We shall see, however that, in addition to being *revisable*, his rule is even *reversible*. It allows for a reversal of strict preference order, if a huge stock of new experimental data points in that direction. The disadvantage of a reversible rule might be that it is inadequate for decision making.

The second point of view is based on epistemological considerations mentioned in the first chapter (p. 3). The very point of being a realist is that it is the only sensible way to explain the lasting success of science. If the rule—no matter the evidence—allows new evidence to reverse the preference order, it cannot explain the success of science. Let φ and ψ be competing theories, and let evidence e favour ψ. A realist explains this preference by the argument that ψ is more similar to the truth than φ. If, subsequently, new evidence reverses this order, then the assumption that φ is more similar to the truth than ψ must explain this second situation; and that would contradict the preceding truthlikeness claim. This is the argument of Laudan and Kuipers agrees.[5] He accepts Laudan's challenge to design a functional or "non-frustrating" rule. The drawback of such an irreversible rule is that it is inadequate as far as scientific practice is concerned. There, as the history of the theory of light shows, preference relations may reverse in the light of new evidence. Our introduction of the rules will consider the question about the *order-fixing elements*; and subsection 4.5.3 returns to the irreversibility of the rules.

4.1.3. Complete Truth

All approach-to-the-truth proposals, except those of Kuipers, are accompanied by the assumption of the completeness of the truth; and even a modal paraphrase of Kuipers's proposals fulfils this assumption. As to the rule of theory-choice, however, this presupposition needs some reflection. At first sight, the complete truth assumption seems harmless. Being innocuous for the definition, why should it hamper the answer to the epistemic problem? Some more reflection, however, reveals that the rule must provide an answer, for all pairs of theories, even if formulated in an indeterminate conceptual framework without a complete truth. After all, due to theoretical terms, there may be epistemic contexts in which the truth of the combination of two languages is incomplete, e.g. a combination of ether and phlogiston. Thus, some propositions may lack a definite truth-value. Although Niiniluoto extends his *definition* such that it handles indeterminate languages he does not elaborate his *rule* in the same direction.[6]

Why, one might ask, must an epistemological rule decide between theories if the truth is indefinite? Suppose we live in the age of phlogiston, and we know that the bulk of observations corroborates or confirms the phlogiston theory. Only a few anomalies occur, but the phlogiston theory is superior to its predecessor. Consequently, we conclude that it "has more truth in it" than previous theories about combustion. We are convinced that new evidence will not reverse the present preference order, and hopefully new refinements, such as the negative weight of phlogiston, will bring us again closer to the truth; but what is the content of the true theory in the phlogiston framework? As far as we know now, disappointingly meagre. It primarily claims that phlogiston does not exist. As the subject of "phlogiston has negative weight" lacks a referent, the sentence does not have a definite truth-value. The rule of theory-choice, however, must order two theories within the phlogiston-oxygen framework where some sentences lack a truth-value. Thus, our conclusion runs as follows. In contrast to the context of the definition, in the epistemic context an adequate rule of theory-choice must inevitably deal with indeterminate languages. If a rule of theory-choice accompanies a truthlikeness definition, it must cope with the situation in which the truth is incomplete. Recently, Niiniluoto has elaborated his point of view regarding meaning variance and truthlikeness.[7]

I end this section with a short remark about concept formation. Strictly speaking, it is not the epistemic rule that has to solve the problem of language variation. In the comparative case, the question whether new evidence has to lead to a new conceptual framework rather than to the formulation of a new theory in the old framework, is beyond the scope of the rule. The rules allow us to choose between two or more existing theories, which together settle the conceptual space; the rule is rather part of the context of justification, than of the context of discovery. How new evidence leads to theories using new concepts is an interesting question that must be kept for another occasion. Anyway, it is rather unlikely that an arbitrary combination of two frameworks always yields a complete truth; and that rules based on a complete truth will not be generally applicable.

4.2. POPPER

Popper's methodological recommendations about falsification precede his ideas about verisimilitude. The former appeared as early as 1935, the year in which *Logik der Forschung* appeared, whereas the latter developed after Popper's conversation with Tarski about Tarski's truth definition and appeared in *Conjectures and Refutations* (1963). The question arises whether these notions match.

First we deal with Popper's own answer. Popper's somewhat rhetorical formulation of the epistemic problem reads: "how do you know that the theory t_2 has a higher degree of verisimilitude than the theory t_1?" His answer is "I do *not* know—I only guess. But I can examine my guess critically, and if it withstands severe criticism, then this fact may be taken as a good critical reason in favour of it (his italics)."[8] According to Popper the claim "t_2 is more verisimilar than t_1" is falsifiable.

Niiniluoto formulates the second possible answer to the question whether the ideas of falsification and verisimilitude match; his answer is negative.[9] He correctly observes that, according to Popper's degree of corroboration, all falsified theories receive the minimum value -1.[10] Consequently, it cannot distinguish between false but verisimilar propositions and false propositions that are not similar to the truth. Popper, however, maintained that "even after t_2 has been refuted in its turn, we can still say that it is better than t_1".[11] Since Popper's intuitions sometimes make more sense then his formal explications, it may be worthwhile to reexamine his ideas about falsification and verisimilitude.

4.2.1. Falsification and Corroboration

As Popper's original verisimilitude fails, we consider the possible relation between Popper's methodological ideas, and his revised notion of verisimilitude. Let us assume that evidence e adequately describes relevant experiments. Then, roughly speaking, Popper provides two methodological rules.

1. Propose a strong theory φ explaining e, and try to falsify it. If the falsification succeeds, try another serious theory explaining the new experimental result e'. Else, if theory φ withstands all serious attempts to overthrow it, e' ($\vDash e$) corroborates φ.
2. For the time being, we accept φ as a sensible explanation of e'.[12]

It is a standard exercise to explain why this methodological process might bring us closer to the truth. The reason is that the falsification of theories, increases the number of crucial experiments. An experiment e is a *crucial test for* φ *and* ψ iff, the truth of e implies the falsification of φ and corroboration of ψ.

1. To find new evidence that falsifies a serious theory φ and explains all previous evidence is making a small, but definite step towards the truth. Let e be the evidence explained by theory φ such that $\varphi \vDash e$, and let a new experiment e' falsify φ. Regarding this new evidence, $e' \vDash e \vDash \neg\varphi$ obtains; apparently, the theory φ has brought us new knowledge $e'$and wept out some failures, false consequences of φ. Suppose we try a new conjecture φ', and it also fails to pass a newly designed experiment; then, again, we have grasped some new knowledge e''. If the experimental data are correct, this accumulation of evidence brings us small steps closer to the truth.

2. Imagine we do not succeed in falsifying a third theory ψ which implies e''. If we were certain about the truth of ψ, we would have made an enormous step towards the truth. There is, however, no certainty about the truth-value of ψ despite our efforts to falsify it. After many failures to falsify ψ, we shall accept ψ and carry on as if it were true. If eventually ψ turns out to be false, we must resume with 1. Meanwhile, we have enriched our knowledge with lots of empirical data.

Figure 1 depicts the situation. Our task is to find the model τ that corresponds to the true theory. The shaded areas certainly do not contain τ, and all our evidence corroborates the theory that τ is an element of ψ and therefore that ψ is true.[13] In short: given error free evidence, crucial experiments bring us small steps closer to the truth. Moreover, failing attempts to falsify a strong theory φ corroborate a larger step towards the truth; if φ is true we have made a substantial step towards the truth.

4.2.2. Why Falsify?

Reading the preceding paragraphs one might object that as long as the falsification succeeds, it makes no difference whether the falsified theory is weak or strong. The reason is that the negation of the evidence is logically stronger than the negation of the theory. If φ implies e, and e has been falsified, φ must be false by virtue of modus tolendo tollens. So why bother trying to formulate strong hypotheses? The answer has two parts. In the first place, strong hypotheses are riskier than weak ones and are more likely to provide crucial experiments; doing so, they give a deeper insight into the structure of nature. In the second place, assuming the truth of two hypotheses, according to the consequence definition, the stronger is closer to the truth—it has more true consequences. If we do not succeed in falsifying a strong theory, it has a reasonable chance to be closer to the truth than the weaker theory.

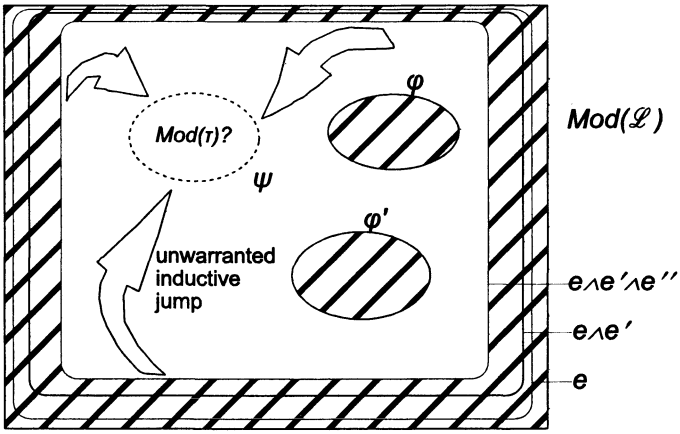


Fig. 1. Falsification and verisimilitude

Popper always stressed the importance of improbable hypotheses. Apparently, this is not a goal for its own sake, but it counterbalances the quest for hypotheses with high probability. We saw (Chapter 1) that the strongest of two true hypotheses is the closest to the truth. A rule of theory-choice based on only high probabilities favours weak summaries of data to risky hypotheses offering new and interesting predictions. A more realistic assessment of theories must balance probability with approach to the truth as, according to the content definition, higher a posteriori probability implies lower verisimilitude. Thus, the degree of the acceptance of φ, $Acc(\varphi)$, must be a function of the posterior probability of a theory and its distance to the truth.

$$Acc(\varphi) := F[P(\varphi \mid e), \mathrm{D}(\varphi,\tau)]$$

Of course, we do not know the actual distance between a theory and the truth. Provided however, that two theories are equally well corroborated, Popper would give the advice to accept the stronger of the two. It is shortsighted to prefer more probable theories to less probable ones without considering their logical strength.

4.2.3. Comments

The order-fixing elements of the consequence definition are logical consequences. The theory with more true consequences is closer to the truth. According to Popper's methodological rules the order-fixing elements are also logical consequences. If two theories are equally well corroborated, again dominance of true consequences makes Popper prefer the stronger one. Therefore, we may conclude that the rule matches the definition.

According to Popper, acceptance of scientific theories is a fallible affair. Moreover, we saw that the conjecture "t_2 is closer to the truth than t_1" is falsifiable as long as t_2 remains corroborated. Thus, Popper's methodological recommendations only provide a fallible indication for the verisimilitude of theories. On the other hand, as all falsified theories receive a degree of corroboration -1, we may conclude that Popper would oppose a reversible rule of theory-choice. Once t_1 has been falsified by crucial experiments that corroborated t_2 Popper will refuse to prefer t_1 to t_2.

Finally, as to the completeness of the truth, Popper (and Miller) do not deal with situations in which the truth is incomplete. They assume the language to be semantically determinate. The complete truth assumption, however, is not an essential part of the content definition. As the work of Kuipers shows, the Δ-proposal is also applicable in situations where the truth is not complete, and the rule of falsification and corroboration does not depend on the completeness of the truth. Therefore, the +-definition, Popper's suggestion to revise his first proposal, and Popper's methodological recommendations suit scientific practice.

4.3. KUIPERS'S STRUCTURALIST APPROACH

Both of Kuipers's comparative approach-to-the-truth proposals are accompanied by rules of theory-choice. As he has given both content and likeness definitions, we shall see whether this distinction also applies to his rules of theory-choice.

4.3.1. Preliminaries

In this subsection we discuss the effects of the structuralist theory representation on Kuipers's answer to the epistemic problem. To that end, I have to summarize the relevant components of the two-tiered structuralist theory representation introduced in Chapter 2. To start with, recall that the structuralist conceptual framework M_p is a class of set theoretical. The naive structuralist representation of a theory X, within this context of relevant concepts consists of two parts.

1. A *conceptual* part: this is the subset $X \subseteq M_p$ of logical possibilities that according to the theory, are physically possible; X is a set theoretical predicate.
2. The *application*: this is the intentional demarcation of the set of real world applications of X. The truth $T = M_p(I_X) \subseteq M_p$ is the set theoretical representation of I_X in M_p.

The empirical claim of the theory X and a law L read $M_p(I) = X$, and $M_p(I) \subseteq L$, respectively. Recall that two theories can only be competitors in the struggle for being the closest to the truth if the frame-hypothesis holds, and the sets of intended applications are identical—that is $Y \leq^{\Delta}_{T} X$ presupposes $I_X = I_Y = I$ and $M_p(I) = T$.

Kuipers's structuralist theory representation has the following intuitive foundation. M_p represent all the logically possible set theoretical structures; and the elements of I_X are all relevant physical situations in the 'real' world. Consider the example in which I_X represents all situations of colliding billiard balls.[14] Then, $M_p(I_X)$ is the set of actual values of the function used to define M_p—such as velocity, position, and force, etc.; and X is the set of function values predicted by the theory X. The strong empirical claim says that the function values of X coincide with the actual values.

Traditional philosophers of science use a formal language to represent scientific theories which are paraphrased by deductively closed sets of $\mathcal{L}$-sentences. The choice of the language then settles two questions.

1. By fixing the situations in the world that are the subject of the theories; it settles the *intended applications of the theories*.
2. By establishing the circumstances under which the theory is true, it settles the *truth conditions for the theories*.

Question 1. roughly corresponds to demarcation of the structuralist set of intended applications and 2. corresponds to fixing the subset X in the structuralist sense. The semantical differences between the received view and the structuralist approach will become important in the sequel.

Since $\mathcal{L}$ settles the intended applications and the truth conditions, model-theoretically, ψ is stronger than φ iff $\psi \vDash \varphi$ (= Mod(ψ) $\subseteq$ Mod(φ)). To highlight the difference with the structuralist situation, let us restrict ourselves to finite propositional languages. Then, a complete theory describes one model (or a set of isomorphic models), and only the contradiction is stronger. The structuralist claim, however, that the theory Y is stronger than X is ambiguous. Generally, it means

$$\text{Mod}(Y) \subseteq \text{Mod}(X) \text{ and } I_X \subseteq I_Y$$

If we leave out the case in which Mod(Y) = Mod(X) and $I_X = I_Y$, it provides at least three different situations to consider. For instance, the theory Y is stronger than the theory X if Mod(Y) = Mod(X) and $I_X \subset I_Y$, a situation that cannot be expressed model theoretically. Even two M_p-complete theories are of different strength if they have different intended applications. For instance, the claim "it is hot, rainy and windy" with intended application Paris, is weaker than the same claim with intended application France.

The ambiguity of the "structuralist strength" of a theory may lead to the following confusion. If we identify a theory with a subset X of M_p-structures, an extension of X and a restriction of X may both imply a stronger theory. *Restriction* of X implies strengthening of X if its elements are considered to be *model theoretical structures*. For example, "it is hot, rainy and windy" is stronger than "it is hot and rainy." *Extension of* X, brought about by increase of I_X and the strong claim $M_p(I_X) = X$, also denotes increase of logical strength. Increase of the domain of application also means increase of its logical strength. For example, applying "it is hot and rainy" to France produces a stronger claim than applying the same predicate to Paris.

The preceding considerations show that we must distinguish between *conceptual strengthening* on the one hand, and *strengthening by application* on the other. The first denotes that the successor theory deductively implies its predecessor; and the second means that the new set of intended applications contains the old one. Conceptual strengthening comes down to excluding elements of $M_p - T$, the logical possibilities that are physically impossible. Strengthening by application denotes the expansion of I_X and therefore of $M_p(I_X)$. We shall see that these two modes of increase of logical strength plays an important role in Kuipers's answer to the epistemological approach-to-the-truth problem.

4.3.2. Kuipers's Success Rules

From 1982 onwards, Kuipers's content and likeness proposals included rules of theory-choice. I shall introduce Kuipers's qualitative, naive quantitative (content), and refined (likeness) rule. Finally, I shall explain why we did not succeed in formulating a reduction proposition for the quantitative likeness rule of theory-choice.

4.3.2.1 The Comparative Content Rule

To emphasize the parallel between Kuipers's definition and his rule, first, I repeat the essentials of Kuipers's content definition of Chapter 2.

Reminder: Let us consider two theories $X \subseteq M_p$ and $Y \subseteq M_p$. According to the Δ-definition, Y is at least as close to the truth as X, iff $Y \Delta T \subseteq X \Delta T$. The definiens consists of two requirements:

1. $T - Y \subseteq T - X$

2. $Y - T \subseteq X(-T)$.

Kuipers calls 1. the *instantial* clause. It requires that if X is right about a situation in the world, then Y should also be right about that situation. Recall that Kuipers interprets the truth as $M_p(I)$, and therefore, the instantial clause embodies strengthening by *application*. The second clause is the *explanatory* part, since it requires that all true laws that X explains, are also explained by Y. In other words, the Cn-clause exemplifies *logical* strengthening of Y.

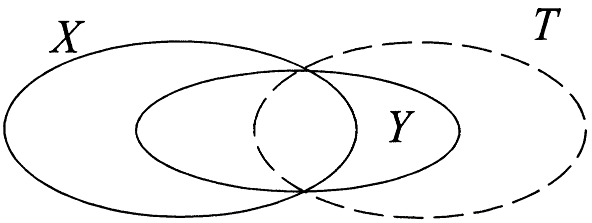


Fig. 2. $Y \leq_T^{Kn} X$

Figure 2 epitomizes Kuipers's naive definition. It illustrates that X and Y have to fulfil strict comparability conditions if they are to be ordered by the naive definition. Using Kuipers's terminology, all mistakes of Y must be mistakes of X, and all truths of X must be truths of Y. *End Reminder*

Now that we have had a reminder of the Δ-definition, let us turn to Kuipers's rule of theory-choice. As noted before, in scientific practice, the expression "Y is at least as close to the truth T as X" must be evaluated on basis of the available evidence. Kuipers solves this problem about theory-choice by relating his verisimilitude definition with "the empirical success of theories." He represents the empirical evidence by two sets. The first set, $R(t) \subseteq M_p$, represents the *experimental data* collected so far. It denotes the set of all physical possibilities realized until time t, $R(t) \subseteq I$. The second set, $S(t) \subseteq M_p$, represents the *strongest law*, or conjunctions of laws that are well corroborated by $R(t)$.[15]

Kuipers's second preparatory step is to assume that the data contain no mistakes. This assumption consists of two parts. In the first place, the instantial evidence $R(t)$ is error free, and therefore $R(t) \subseteq M_p(I) = T$. In the second place, the strongest accepted law, $S(t)$, is supposedly true; thus $T \subseteq S(t)$. Kuipers bases his rule of success on the conjunction of these two assumptions, which he calls the *correct-data hypothesis*:

(3) $\quad R(t) \subseteq T \subseteq S(t)$

With a correct $R(t)$ and $S(t)$ at his disposal, Kuipers defines Y's success predominance as follows:[16]

DEFINITION 4.2: The theory Y is *at least as successful* as X, $NMS(X,Y,R/S)$, iff
1. $R(t)\text{-}Y \subseteq (R(t))\text{-}X$,
2. $Y\text{-}S(t) \subseteq X(\text{-}S(t))$

Notation: $Y <^{\Delta}_{R/S} X$

Note the parallelism between the content definition and definition 4.2. The second consists of (1.) an instantial clause, and (2.) an explanatory clause. The theory Y is at least as successful as X if and only if Y contains all $R(t)$-instances of X, and X contains all $S(t)$-problems of Y. Figure 3 epitomizes definition 4.2. It shows that $Y <^{\Delta}_{R/S} X$ may also be paraphrased by: *Y is closer to the revealed truth than X.* Clearly, the situation in definition 4.2. is not the only pattern of X, Y and T that reflects a surplus of success of Y. For example, if X and Y both explain the strongest law S, which cannot be excluded for serious theories, the explanatory clause is trivially fulfilled.

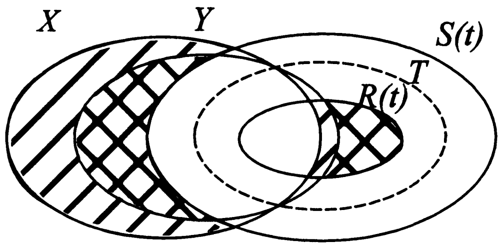


Fig. 3. *NMS(X, Y, R/S)*

Finally, after Kuipers has defined the notion of "being at least as successful" he uses this notion to formulate his *Rule of Success*, it reads:

> "If theory Y is more successful than theory X, then choose theory Y (for the time being)" (*Kuipers 1992 p. 310).*

In other words, if until now, R and S have uniformly favoured the theory Y to X, choose Y until we encounter a crucial experiment in favour of X; then, the two theories become uncomparable as far as the success rule is concerned. The intention of the success rule is clear. If the set of instantial data R increases, the logical strength of the strongest accepted law increases, and the set of S-models decreases; and eventually R and S will approximate the truth. More recently, Kuipers has added to the constraint "Y is more successful than theory X" until now, the requirement that Y must *seem to remain* more successful.[17] Evidently, this raises the problem about indications of lasting empirical success. Furthermore, note that the claim "$Y <^{\Delta}_{R/S} X$" is falsifiable, since there may turn up experimental data that corroborate X and falsify Y. The same claim is not verifiable. It is based on the assumption that T is true, $T \subseteq S(t)$, and although $S(t)$ is the strongest law allowed by the empirical data found up to time t, we may not conclude that it is true. Apparently, the correct-data hypothesis is too strong as far as $S(t)$ is concerned. This will be the subject of the subsection 4.3.3.4.

Let us consider the relation between definition 4.2 and the Δ-definition. The next proposition, formulated and proved by Kuipers, claims that more similarity to the truth implies more empirical success.

PROPOSITION 4.1: Let X, Y and $M_p(I)$ be subsets of M_p. Then, for all R and S satisfying the correct-data hypothesis: $Y \leq^{\Delta}_{T} X$ implies $Y \leq^{\Delta}_{R/S} X$

Proof: Instantial clause: $T-Y \subseteq T-X \Leftrightarrow X \cap (T-Y) \subseteq X \cap (T-X) = \varnothing$, which with $R \subseteq T$, implies $X \cap (R-Y) = \varnothing \Leftrightarrow R-Y \subseteq R-X$. Explanatory clause: $Y-T \subseteq X-T \Leftrightarrow Y-(T \cup X) \subseteq X-(T \cup X) = \varnothing$, with $T \subseteq S$ it implies $Y-(S \cup X) = \varnothing \Leftrightarrow Y-S \subseteq X-S$. ⊠

Intuitively, the proposition claims that, *given* the correct-data hypothesis, predominance of verisimilitude explains a surplus of experimental success. Figure 3 illustrates the theorem. It shows that $T-Y \subseteq T-X$ implies $R-Y \subseteq R-X$, and $Y-T \subseteq X-T$, implies $Y-S \subseteq X-S$. In due course, I shall show that the naive rule of success is a functional rule, viz. $Y <^{\Delta}_{R/S} X$ implies $X \nless^{\Delta}_{T} Y$ (equation (8), p. 139).

4.3.2.2 The Quantitative Content Rule

Kuipers's quantification of his content approach consists of replacing the inclusions of the symmetric differences by comparison of the cardinalities of those differences. We saw in Chapter 2 that according to Kuipers's quantitative naive content definition the theory Y *is at least as close to the truth as* X, iff

(4) $|T-Y| + |Y-T| \leq |T-X| + |X-T|$ which is equal to
$|(X \cap T)-Y| + |Y-(X \cup T)| \leq |Y \cap (T-X)| + |X-(Y \cup T)|$

Let $NTD(X \backslash T)$ designate $|X-T|$ and let $NTD(Y,R/S)$ refer to $NTD(R(t)\backslash Y) + NTD(Y \backslash S(t))$; then the quantitative counterpart of the naive qualitative success rule reads as follows:

DEFINITION 4.3: The theory Y is *naive quantitatively at least as successful* as X iff $NTD(Y, R/S) \leq NTD(X, R\backslash S)$
Notation: $Y \leq^{Dn}_{R/S} \mathrm{X}$

In terms of cardinalities the definition reads:

(5) $|X \cap (R-Y)| + |Y-(X \cup S)| \leq |Y \cap (R-X)| + |X-(Y \cup S)|$

Again, we may distinguish between an instantial clause, the R-part of definition 4.3, and an explanatory clause, the S-part. One consequence of the definition is that for all X with $R \subseteq X \subseteq S$, trivially $NTD(X,R\backslash S) = 0$, and therefore $NTD(R,R\backslash S) = NTD(T,R\backslash S) = NTD(S,R\backslash S) = 0$. In other words, all theories X with $R \subseteq X \subseteq S$ are equally successful and their distance from the data is zero. Moreover, there is no analogue to proposition 4.1, and the naive *quantitative* approach does not provide a functional rule of success.

PROPOSITION 4.2: Let the correct-data hypothesis obtain for some R and S; then $Y <^{Dn}_{R/S} X$ does excludes $X <^{Dn}_{T} Y$

Proof: It suffices to construct a counter example. Let $|(X \cap R) - Y| + |Y - (S \cup X)| = a$, $|(Y \cap R) - X| + |X - (S \cup Y)| = 2a$, $|(X \cap T) - Y| + |Y - (T \cup X)| = 4a$, and $|(Y \cap T) - X| + |X - (T \cup Y)| = 3a$. Then, consistently $Y <^{Dn}_{R/S} X$, and $X <^{Dn}_{T} Y$. ⊠

4.3.2.3 The Comparative Likeness Rule

Parallel to the content proposal, in the likeness approach Kuipers grafts his rule onto the definition. To facilitate the comparison of rule and definition I will repeat the definition.

Reminder: Let X, Y and T be subsets of M_p; According to Kuipers's comparative likeness definition, Y is at least as close to the truth as X iff

1. for all x in X and t in T if $r(x, t)$, then there is a y in Y such that $s(x,y,t)$
2. for all y in $Y - (X \cup T)$ there are x in $X - T$, and t in T such that $s(x,y,t)$

$s(x,y,t)$ abbreviates that y is at least as similar to t as x (the relation need not be symmetric), and $r(x,t)$ means that x *is comparable with t.* The r-relation is defined in terms of $s(x,y,t)$: $r(x,t)$ obtains if there is a y in M_p such that $s(x,y,t)$. Recall that r need not be symmetric nor transitive, since the underlying s-relation may lack these properties. Generally, Kuipers does not add the constraint that theories must be convex. Then, 1. is the instantial, and 2. the explanatory clause. If, however, we add the convexity constraint, the first clause concerns the best elements of the rival theories, and the second one involves the worst elements. *End Reminder*

Similar to the content situation, in the likeness case, Kuipers represents the evidence by two sets; $R(t)$ is the set of instances, and $S(t)$ is the set of M_p-structures representing the strongest accepted law. Assuming correct data, $R(t) \subseteq T \subseteq S(t)$, Kuipers grafts his refined rule onto his refined definition:[18]

DEFINITION 4.4: *The theory Y* is, w.r.t. $R(t)$, $S(t)$, *at least as successful as* X iff

1. For all x in X and z in $R(t)$ if $r(x, z)$, then there is a y in Y such that $s(x,y,z)$
2. For all y in $Y-(X \cup S(t))$ there are x in X-$S(t)$ and z in $S(t)$ such that $s(x,y,z)$

Notation: $Y \leq^{Kr}_{R/S} X$

The parallelism between rule and definition is straightforward. The instantial clause of the rule equals the instantial clause of the definition if we substitute $R(t)$ for T; if we substitute $S(t)$ for T, also the explanatory clauses of the rule and the definition are the same.

We saw (prop. 4.1, p. 133) that regarding the *content proposal*, the instantial clause of the definition implies the instantial clause of the rule; and the same holds for the explanatory clause. As for the *likeness approach*, the instantial clause of the definition also implies the instantial clause of the rule. Regarding the explanatory clause, however, if the theories and $S(t)$ are not convex, the likeness definition does not imply the explanatory clause of the rule. If for all y in $Y-(X \cup T)$ there are x in $X-T$ and z in T such that $s(x,y,z)$, then $T \subseteq S(t)$ does not exclude that there is a $y \in Y-(X \cup S(t))$ such that there is no x in $X-S(t)$ and z in $S(t)$ that $s(x,y,z)$. The reason is that $S(t) \supseteq T$ and the $x \in X-T$ such that $s(x,y,z)$ need not be in $X-S(t)$. Thus, regarding the likeness approach, closer to the truth does not straightforwardly imply being more successful. Kuipers showed, however, that for convex X, Y and $S(t)$, $Y \leq^{Kr}_{T} X$, implies $Y \leq^{Kr}_{R/S} X$.[19]

PROPOSITION 4.3: Let the correct data hypothesis apply for arbitrary $R, S \subseteq M_p$ and let X, Y, T, and $S \subseteq M_p$ be convex. Then: $Y \leq^{Kr}_{T} X$, implies $Y \leq^{Kr}_{R/S} X$

Proof: 1. $Y \leq^{Kr}_{T} X$ trivially implies $Y \leq^{Kr}_{R/-} X$. 2. We must prove that $Y \leq^{Kr}_{T} X$ implies $Y \leq^{Kr}_{-/S} X$. Let y be an arbitrary y in $Y-(X \cup S(t))$. Then there is an x in $X-T$ and a z in $T \subseteq S(t)$ such that $s(x,y,z)$. The implication obtains, if $x \in X-S(t)$. If $x \in X-(T \cap S(t))$ then $x \in S(t)$ and $z \in S(t)$, but $y \notin S(t)$ and $s(x,y,z)$. Hence, $S(t)$ is not convex. This contradicts the assumption, and therefore x is an element of $X-S(t)$. ⊠

Corollary 4: Let the circumstances of the preceding proposition hold; then, $<^{Kr}_{R/S}$ is functional for approaching the truth; that is to say: $Y <^{Kr}_{R/S} X$ implies $X \nless^{Kr}_{T} Y$.
Proof: Suppose $Y <^{Kr}_{R/S} X$; this implies by definition $X \nleq^{Kr}_{R/S} Y$, which by the preceding proposition implies $X \nleq^{Kr}_{T} Y$. Consequently, $X \nless^{Kr}_{T} Y$. ⊠

As Kuipers's likeness success rule is grafted onto his likeness definition, the rule has inherited the likeness character of the definition. Figure 5 shows the situation in which all M_p-elements are comparable (for directed r) and the theories are convex. S contains the strongest truth T and the dots in T represent the (non-convex) subset R. The diagram suggests that the singleton with the worst element of M_p is the worst theory.

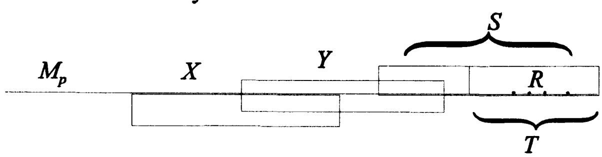


Fig. 4. *An RTL(X, Y, R/S) example*

Regarding formal languages, for any pair S and R respecting the correct data hypothesis, all theories are better than the complete falsehood ξ, which is considerably worse than the negation of the truth. The next proposition concerns the situation in which all $x \in M_p$ are comparable to ξ, the worst element of M_p. It says that $\{\xi\}$ is the worst theory, if ξ be the worst element of M_p regarding s and T.

PROPOSITION 4.5: Let $T \subseteq M_p$, and let for all $y \in M_p$ and all $z \in T$, $s(\xi,y,z)$. Then, for all nonempty Y, $R(t)$, $S(t) \subseteq M_p$ with $R(t) \subseteq T \subseteq S(t)$: $Y \leq^{Kr}_{R/S} \{\xi\}$

Proof: By definition we have to prove: 1. For ξ in $\{\xi\}$ and all z in $R(t)$ if $r(\xi, z)$, then there is a y in Y such that $s(\xi,y,z)$; 2. For all y in $Y-(\{\xi\} \cup S(t))$ and ξ in $\{\xi\}$-$S(t)$, $\exists z$ in $S(t)$ such that $s(\xi, y, z)$. Ad 1. As $Y \neq \varnothing$, $R(t) \subseteq T$, and for ξ, $\forall y \in M_p$, $\forall z \in T$: $s(\xi, y,z)$, it follows that for ξ and $\forall z \in R(t)$ if $r(\xi, z)$, then $\exists y \in Y$: $s(\xi,y,z)$. 2. As $\forall y \in M_p$, $\forall z \in T$, $s(\xi,y,z)$, it follows that $\forall y \in Y-(\{\xi\}-S(t))$, $\exists z \in S(t)$ with $s(\xi,y,z)$. This proves 2. if $\xi \notin S(t)$; but if $\xi \in S(t)$, then $Y-(\{\xi\} \cup S(t)) = \varnothing$, and 2. is trivially fulfilled. ⊠

Summarizing the present subsection, we saw that Kuipers has grafted his refined rule onto his refined definition; therefore the first rule has inherited the likeness character of the second. Without additional constraints, however, the rule is not functional for approaching the truth. Kuipers proved that convexity of the theories restores this functionality.

4.3.2.4 The Quantitative Likeness Rule

This final subsection concerns Kuipers's proposal to define a quantitative version of his likeness rule. This proposal matches closely Kuipers's quantitative refined definition of the preceding chapter. I shall introduce the instantial and explanatory clause separately.

Let (M_p, d) be a metric space as defined in Chapter 3, $D(X\backslash Y)$ denotes $\sum_{x \in X} d(x,Y)$, and $d(x,Y)$ is equal to $\min\{d(x,y) \mid y \in Y\}$. Then, Kuipers defines the *instantial* (refined) quantitative success of a theory as follows:

DEFINITION 4.5: The theory Y is *instantially more successful* (w.r.t. R) than X iff $D(R\backslash Y) < D(R\backslash X)$

In a unpublished report Kuipers observes that $D(R\backslash X)$ has the following three properties:

1. $0 = D(\varnothing\backslash X) \leq D(R\backslash X) \leq D(T\backslash X)$
2. $D(R\backslash X) = 0$ iff $R \subseteq X$
3. $D(R\text{-}X\backslash X) = D(R\backslash X)$

The third property shows that $R \cap X$ does not contribute to $D(R \backslash X)$, and $D(R \backslash X)$ only takes the R-mistakes of X into account. According to Kuipers, the *explanatory success* is the extent to which a theory succeeds in explaining supposedly true laws.

DEFINITION 4.6: The theory Y is *explanatory at least as successful* (w.r.t. S) as X iff $D(X \backslash S) \geq D(Y \backslash S)$

Analogous to the instantial part, Kuipers introduced the following properties of $D(X \backslash S)$.

1. $0 = D(X \backslash M_p) \leq D(X \backslash S) \leq D(X \backslash T)$
2. $D(X \backslash S) = 0$ iff $X \subseteq S$
3. $D(X\text{-}S \backslash S) = D(X \backslash S)$

Again, it should be noted that $S \cap X$ does not contribute to $D(X \backslash S)$. According to the definitions, only the *S-mistakes* of X and the *X-mistakes* of S determine the distance between X and S.

Next, Kuipers defines the refined quantitative distance between the theory X and data R/S, using an instantial part $D(R \backslash X)$ and an explanatory part $D(X \backslash S)$: $D(X,R \backslash S) := D(R \backslash X) + D(X \backslash S)$. Finally, he fixes the meaning of "theory Y is more successful than X:"

DEFINITION 4.7: The theory Y is *at least as successful* (w.r.t. R/S) *as* X iff $D(X,R \backslash S) \geq D(Y,R \backslash S)$
Notation: $Y \leq^{D}_{R/S} X$

Although the preceding definitions are plausible, they do not yield results similar to the comparative refined proposal. Without further conditions and modifications it is certainly impossible to prove the basic theorem analogous to the one in the comparative case.

(6) $\quad \mathrm{D}(X,T) \geq \mathrm{D}(Y,T) \overset{?}{\Rightarrow} \mathrm{D}(X,R/S) \geq \mathrm{D}(Y,R/S)$ (for all $R \subseteq T \subseteq S$)

This statement need not surprise us. As we saw in Chapter 3 the averaging methods of the comparative proposal and the quantitative one are different. Here the averages between theories and data fix the overall distances whereas in the comparative case only the extremes settle the distances. Thus, one extreme member of T may be the cause for the sum of T to X to be bigger than the sum of T to Y. Leaving out this extreme T member, X might be closer to the rest of T and the set of models representing the empirical evidence R.

PROPOSITION 4.6: $\mathrm{D}(X,T) > \mathrm{D}(Y,T)$ does not exclude that there is a R, S-pair with $R \subseteq T \subseteq S$ such that $\mathrm{D}(X,R/S) < \mathrm{D}(Y,R/S)$

EXAMPLE: Let $X := \{x_1, x_2, x_3\}$, $Y := \{y_1, y_2, y_3\}$, $T := \{t_1, t_2, t_3\}$, $R := \{t_1, t_2\}$, and $S := X \cup Y \cup T$. Next, suppose that the smallest distances from an element to a set is "along the same subscript"—that is $d(x_i, T) = d(x_i, t_i)$ for $i = 1, 2, 3$. Let the distances be defined by $d(x_i, t_i) = i^{i-1}$, $d(y_i, t_i) = i + {}^1/_i$ with $i = 1, 2, 3$. Then, from $D(T \backslash X) > D(T \backslash Y)$ and $D(R \backslash X) \ngtr D(R \backslash Y)$, ($12 > 7^5/_6$ and $3 \ngtr 4^1/_2$), follows $D(X,T) > D(Y,T)$ and since the distances towards S equal 0, it also follows $D(X,R/S) < D(Y,R/S)$. *End Example*

Nevertheless, perhaps we can prove a weaker version of implication (6). First, I shall consider the instantial clauses. Let T_r be defined by $\{R \mid R \subseteq T, 0 < |R| = r \leq |T|\}$). Then, Kuipers has considered the notion "average distance of r-sized subsets R of T:"

DEFINITION 4.8: $\overline{d_r(T/X)} :=_{def} \sum_{R \in T_r} \frac{D(R \backslash X)}{|T_r|}$

Next, Kuipers proved that $\Sigma_{R \in T_r} D(R \backslash X)$ must be proportional to $D(T,X)$.[20]

PROPOSITION 4.7: $D(T \backslash X) \geq D(T \backslash Y) \Rightarrow \overline{d_r(R/X)} \geq \overline{d_r(R/Y)}$

Proof: Let $D(T \backslash X) \geq D(T \backslash Y)$. Take an arbitrary element $t \in T$. If $0 < r < |T|$, then there are exactly $\frac{(|T|-1)!}{(r-1)!\,(|T|-r)!}$ possible r-sized subsets of T that contain t. Call this number a. We want to calculate the $\Sigma_{R \in T_r} D(R \backslash X)$. As we did not assume anything special about t, every $t \in T$ contributes $a \times d(t,X)$ to $\Sigma_{R \in T_r} D(R \backslash X)$. Hence $\Sigma_{R \in T_r} D(R \backslash X) = \Sigma_{t \in T}\, a \times d(t,X) = a \times \Sigma_{t \in T}\, d(t,X) = a \times D(T \backslash X)$. Equally $\Sigma_{R \in T_r} D(R \backslash Y) = a \times D(T \backslash Y)$ and because $a \geq 1$, we have $\sum_{R \in T_r} \frac{D(R \backslash X)}{|T_r|} \geq \sum_{R \in T_r} \frac{D(R \backslash Y)}{|T_r|}$ ⊠

The proposition says that if Y is quantitatively refined, instantially more truthlike than X, generally it will have more instantial empirical success. Consequently, the hypothesis that the distance from the truth to Y is smaller than that from X to the truth explains that the distance from the instantial r-sized evidence R to Y is smaller than the distance between R to X. For, if the first holds most r-sized subsets R are closer to Y than to X.

As to the explanatory clause, we might construct an analogous argument, if we define an $S_{max} := X \cup Y \cup T$ and define $T_s := \{S \mid T \subseteq S \subseteq S_{max}\ |S| = s\}$. The proposition is, however, dependent on X and Y, which has to be avoided. Despite our efforts, we did not manage to come up with a reasonable alternative.

4.3.3. Comments and Observations

This subsection concerns two unnoticed implications of the success rules, problems with the assumptions of the correct data hypothesis and convexity. I shall also analyse the problems with the refined quantitative version of the success rule.

4.3.3.1 X closer to the truth; and Y equally successful

In the preceding subsections, I showed that the comparative refined order of the success-rules are not antisymmetric; $Y \leq^{\Delta}_{R/S} X$ and $X \leq^{\Delta}_{R/S} Y$ do not imply $X = Y$. This lack of antisymmetry brings along a slight complication for the naive and refined rule. Kuipers's claims that $Y \leq^{\Delta}_{R/S} X$ can be *explained* by the conjunction of the truth-approximation hypothesis $Y <_T X$ and the correct-data hypothesis. Strictly speaking, this is only half of the truth.[21] The claim surpasses the fact that if, besides $Y \leq^{\Delta}_{R/S} X$, $X \leq^{\Delta}_{R/S} Y$ also holds, $X <_T Y$ rather than $Y <_T X$, or even $Y \not\asymp_T X$, may obtain. In other words,

(7) $\quad Y \leq^{\Delta}_{R/S} X \not\Rightarrow \neg(X <_T Y)$,

and $Y \leq^{\Delta}_{R/S}$ X and $X <_T Y$ are not mutually exclusive. For example, let $X-(T \cup Y) = Y-(S \cup X) = (R \cap X)-Y = (T \cap Y)-X = \varnothing$ (see Figure 6).

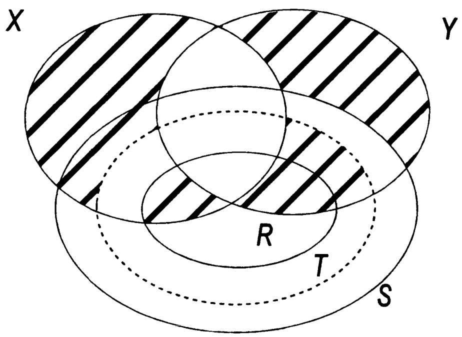


Fig. 6. $Y \leq_{R/S} X$ *and* $X <_T Y$

Then, Y is at least as close to the data as X, $Y \leq^{\Delta}_{R/S}$ X, and X is closer to the truth than Y, $X <_T Y$. Consequently, Kuipers's comparative content definition *functionally approaches* the truth in the following sense (see p. 123):

(8) $\quad Y <^{\Delta}_{R/S} X \Rightarrow \neg(X <^{\Delta}_{T} Y)$

That is, one crucial experiment in favour of Y prevents X from being closer to the truth; another crucial experiment in favour of X renders the theories uncomparable. Following the content definition, the content rule is truth-value dependent.

PROPOSITION 4.8: The naive success rule is *irreversible*. This means: for any $\langle R', S' \rangle$ with $R \subset R'$ and $S' \subset S$: $Y <^{\Delta}_{R/S} X$ implies $X \nless^{\Delta}_{R'/S'} Y$

Proof: Suppose $Y <^{\Delta}_{R/S} X$; then it follows $[(Y \cap R) - X] \cup \{(X - (Y \cup S)] \neq \varnothing$. Since $R \subset R'$ and $S' \subset S$, $[(Y \cap R') - X] \cup \{(X - (Y \cup S')] \neq \varnothing$; thus $X \nless^{\Delta}_{R'/S'} Y$. ⊠

4.3.3.2 Likeness rule and Content Rule

This subsection considers the naive and refined structuralist rules of theory-choice. According to the three preceding chapters, the naive-refined distinction coincides with the content-likeness contrast. Consequently, it is plausible to ask whether, and if so, under what conditions, the naive rule of theory-choice contradicts the refined rule. The following example shows that in certain circumstances the content preference order contradicts the preference order of the likeness rule.

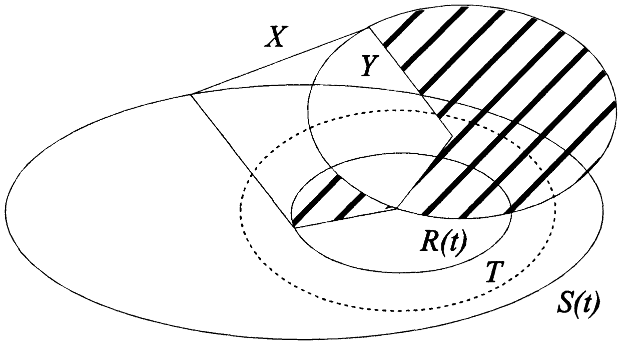
EXAMPLE: Consider the alphabet normally ordered, and the convex truth $T := \{a,b,c\}$. Let us assume that at some time t the non convex $R(t)$ is $\{a, c\}$, and let Y and X be the convex $\{g\}$ and $\{f, g\}$, respectively. Finally, let $S(t)$ be $\{a, b, c, d, e\}$, a convex set containing T. In these circumstances according to the content rule (Kuipers's comparative naive one) Y is at least as successful as X, and X is not at least as successful as Y: $Y <^{\Delta}_{R/S} X$ and Y is more successful than X. Kuipers's likeness rule, however, claims that X is at least as successful as Y, and Y is not at least as successful as X: $X <^{Kr}_{R/S} Y$. That is to say, X is more successful than Y. This example shows that the content rule sometimes contradicts the likeness rule.

End Example

One might object that the example concerns a degenerated case, since the difference between $S(t)$ and $R(t)$ does not count. This is correct, but, the definition allows for this non trivial example. The question arises whether the two rules may contradict when $X \cap R(t) \neq \varnothing$. The answer is positive.

PROPOSITION 4.9: $Y <^{\Delta}_{R/S} X$ (with $X \nless^{\Delta}_{R/S} Y$) does not exclude: $X <^{Kr}_{R/S} Y$ such that $Y \nless^{Kr}_{R/S} X$ and $X \cap R(t) \neq \varnothing$.

EXAMPLE: Let $s(x,y,z)$ designate "y is an element of the Euclidian line between (x, z)." In the next diagram $Y \subset X$, $(X \cap R(t)) - Y = \varnothing$, and finally $X - (Y \cup S(t)) \neq \varnothing$; Consequently, $Y <^{\Delta}_{R/S}$ X, and $X \nless^{\Delta}_{R/S} Y$. Furthermore 1. For all y in Y and z in $R(t)$ if $r(y, z)$, then there is a x in X such that $s(y,x,z)$ (since $X \supset Y$). 2. For all x in $X - (Y \cup S(t))$ there are y in Y-$S(t)$ and z in $S(t)$ such that $s(y,x,z)$; and finally, since there are $x \in X$ and $z \in R(t)$ without any $y \in Y$ such that $s(x,y,z)$, there are lines beginning in X and ending in $R(t)$ without passing through Y, we may conclude that $X <^{Kr}_{R/S} Y$ and $Y \nless^{Kr}_{R/S} X$ hold both; note that all sets are convex. ⊠



4.3.3.3 Two-level correction of the correct data hypothesis

Application of the structuralist success rule presupposes the *correct data hypothesis*. That is, at every time t, $R(t) \subseteq T \subseteq S(t)$. Despite the rather elementary appearance of the assumption, it needs some reflection. As far as the hypothesis assumes that the experiments are error free, $R(t) \subseteq T$, it is straightforward and without complications. There are, however, at least two ways in which the hypothesis can mislead an unwary reader; these are the physical truth of $S(t)$, and the interpretation of the two inclusion signs. I will discuss the physical truth of the strongest accepted law $S(t)$ in subsection 4.3.3.4; and in the present subsection I investigate the interpretation of the two inclusion signs in $R(t) \subseteq T \subseteq S(t)$.

According to Kuipers's interpretation, the first inclusion sign $R(t) \subseteq M_p(I)$, yields an assertion about experimental reports; it denotes the fact that the reports are without writing and calculation errors.[22] The second inclusion sign, $M_p(I) \subseteq S(t)$, is a model theoretical one, designating that *all models of the truth* are also *models of the strongest accepted law*. It produces a conceptual claim, since it claims that T logically implies S. The difference is even more manifest, if, for the sake of the argument, we replace the unknown theory T by some familiar theory X. Then, according to Kuipers, $R(t) \subseteq X \subseteq S(t)$ means: for all the experimental situations in $R(t)$, the theory X *was true*, and X *logically implies* the strongest accepted law $S(t)$. The first statement must be verified experimentally, whereas the second is a strictly logical claim. What happens if we neglect this fundamental difference? For the sake of convenience, in what follows, we will not distinguish between background knowledge and the initial conditions of all experimental data.

To start with, let us assume that M_p consists of propositional constituents. Then, Kuipers claims that

"If Y is a subset of X, ... hypothesis Y is also said to imply hypothesis X, in agreement with standard model-theoretic usage ..." [23]

Consequently, if $R(t) \subseteq T$ and $T \subseteq S(t)$ are *both* model theoretical set inclusions, the experiments, in combination with the initial conditions, *deductively imply* the strongest accepted law S, $R \vDash S$, since $\mathrm{Mod}(R) \subseteq \mathrm{Mod}(S)$. This contradicts the widespread opinion that empirical data that corroborate a law X are represented as *consequences* of X. Thus, the first reason to distinguish the two inclusion signs is to block the consequence of the correct data hypothesis that experiments, combined with the initial conditions, would deductively *imply* the strongest law S.

Second, the argument of logical strength also affects the dynamics of the methodological rule. According to the structuralists, experiments lead to "... new supplementary data, resulting in $R(t) \subseteq R(t') \subseteq T$ and $T \subseteq S(t') \subseteq S(t)$, for t' later than t." [24] As $R(t) \subseteq R(t')$ means that the number of experiments has increased, it denotes the fact that $R(t)$ is logically *weaker* than $R(t')$. If, on the basis of these new experiments, scientists succeed in formulating a stronger, general accepted law $S(t')$, then Kuipers paraphrases this fact by $S(t') \subseteq S(t)$; and $S(t')$ is logically *stronger* than $S(t)$. Since there is no difference between the kind of models of T and of $\neg T$, this leads to the paradoxical situation that $A \subset B$ and $A \supset B$ both mean increase of logical strength.

Furthermore, if $R(t) \subseteq R(t')$, means $\mathrm{Mod}(R(t)) \subseteq \mathrm{Mod}(R(t'))$, then $R(t)$ would logically imply $R(t')$, and the first experiment, $R(t)$ would deductively imply the outcome of the next experiment $R(t')$. This is, of course, unacceptable. Moreover, if R contains more than one model and claims that both are adequate, it even contains a contradiction, since both constituents cannot be true at the same time.[25] My conclusion reads that to avoid these strange consequences of the correct data hypothesis, the interpretation of its two inclusion signs must differ significantly. Consequently, I reformulate the hypothesis to:

(9) $R(t) \subseteq_1 T \subseteq_2 S(t)$

Fortunately, Kuipers's intuitive description of the definitions gives the clue to an adequate interpretation of the correct data hypothesis. The correct interpretation of (9) must reckon with the *different levels of the inclusions*. To clarify the meaning of this contention, I will elaborate an example of a monadic predicate language. This example will be reused to compare Kuipers's refined rule of success with Niiniluoto's rules of estimated truthlikeness in Section 4.5.

EXAMPLE: Let $\mathscr{L}[V(x), W(x)]$ be a monadic predicate language without names for individuals, and with vocabulary $\{V(x), W(x)\}$. It is generally known that these two atomic predicates form the following four Carnapian Q-predicates: $Q_1(x) := V(x) \wedge W(x)$, $Q_2(x) := V(x) \wedge \neg W(x)$, $Q_3(x) := \neg V(x) \wedge W(x)$, $Q_4(x) := \neg V(x) \wedge \neg W(x)$.

Furthermore, all propositions of $\mathscr{L}[V(x), W(x)]$ are disjunctions of constituents with the following form:

$$C_i := \exists x Q_m(x) \wedge \exists x Q_{m+1}(x) \wedge \ldots \wedge \exists x Q_n(x) \wedge \forall x[Q_m(x) \vee Q_{m+1}(x) \vee \ldots \vee Q_n(x)] \text{ with } 1 \leq m \leq n \leq 2^{|\mathrm{voc}(\mathscr{L})|}$$

Leaving out the empty constituent, $\mathscr{L}[V(x), W(x)]$ has $2^4 - 1 = 15$ constituents, and, hence 2^{15} propositions. In the Boolean algebra of the next diagram, the ⊞-boxes systematically represent the constituents of $\mathscr{L}$. The shaded part of a ⊞-box represents the instantiated Q_i-predicates; the other Q_i-predicates are empty. For example, the constituent $\exists x Q_4(x) \wedge \forall x Q_4(x)$ is at the right side at the bottom. Note that all ⊞-boxes are constituents, and the lines between them *do not represent a logical deduction relation* but the smallest Clifford distances between the elements. One constituent represents the complete empirical truth of $\mathscr{L}$. In our example, the truth is represented by the constituent $\exists x Q_1(x) \wedge \exists x Q_2(x) \wedge \exists x Q_3(x) \wedge \forall x[Q_1(x) \vee Q_2(x) \vee Q_3(x)]$.

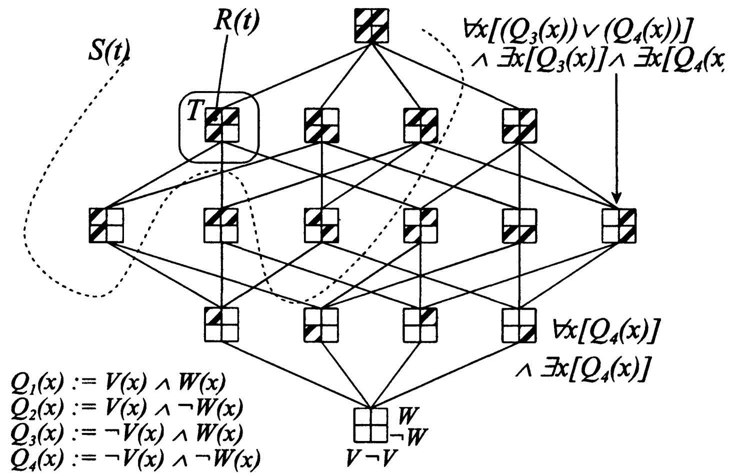


Fig. 8. *The monadic predicate example*

As to the experimental record of this toy example, incoming evidence is represented by instantiations of the Q_i-predicates. Suppose that until time t, scientists only encountered individuals that were V and W. This evidence is represented by the instantiation of Q_1 in T. Suppose further that, on basis of these instantiations the strongest generally accepted law $S(t)$ is the disjunction of all constituents lying at the inside of the dotted line; since $T \in \mathrm{Cnst}(S(t))$, $S(t)$ is true. Under this interpretation, where $R(t)$ is the set of instantiated Q_i-predicates of T, extension of $R(t)$ and contraction of $S(t)$ both mean an increase of logical strength.

End Example

Let us return to the interpretation of the perfect data hypothesis (9). In our monadic predicate example, $R \subseteq_1 T$ means that all *instantiated* Q_i-predicates of $R(t)$ are also instantiated in T. Again, this boils down to the constraint that there are no mistakes in the experimentation records. The second part of the hypothesis does not occur on the level of Q_i-predicates but on the level of $\mathscr{L}$-models (or constituents). Thus $T \subseteq_2 S$, abbreviates that the constituent representing the truth implies S; in other words S is true.

Language $\mathscr{L}$ enables us to express the constraint of Q_i-predicate instantiation in syntactic terms; $R^s := \exists x Q_m(x) \wedge \exists x Q_{m+1}(x) \wedge \ldots \wedge \exists x Q_n(x)$. Therefore, we may formulate *both* set inclusions of the perfect data hypothesis on a syntactic or on a semantical level ($R^m := \mathrm{Mod}(R^s)$):

(10) $R^s \subseteq_1 (S \subseteq_1)\ M_p(I)$, which means $\mathrm{Cn}(R^s) \subseteq (\mathrm{Cn}(S) \subseteq)\ \mathrm{Cn}(T)$
(11) $T \subseteq_2 S\ (\subseteq_2 R^m)$, which means $\mathrm{Mod}(T) \subseteq \mathrm{Mod}(S)\ (\subseteq \mathrm{Mod}(R^s))$

The preceding analysis shows that we must interpret Figure 3 (p.132) carefully. It suggests that the truth is "squeezed in" between $R(t)$ and $S(t)$, just as, in the integral calculus, the upper sum and lower sum approach the area under a function when the partition is refined. Formulae (10) and (11) show that this suggestion is wrong. In both cases, the truth is a lower bound and it is not "squeezed in" from two different directions.

The careful reader might have noticed that the monadic example is analogous to the modal explication of Chapter 2. Under that interpretation, $R \subseteq_1 T$ means that the R-worlds are allowed by the modal constituent representing the truth, and nothing is said about the worlds $w \notin R$. Further, $T \subseteq_2 S$ implies that for all modal models $\mathfrak{M} \notin S$ obtains $\mathfrak{M} \vDash \neg T$, and nothing is said about the $\mathfrak{M} \in S$. In this sense, increase of R and decrease of S both result in conceptual strengthening of R and S (subsection 4.3.4). The monadic predicate example and the modal perspective both show that the inclusions of the correct data hypothesis occur on fundamentally different levels.

Finally note that our two-level paraphrase of Kuipers's correct data hypothesis backfires on the entire structuralist likeness approach. According to our interpretation, the refined rule and definition mix up the conceptual level and the level of application. First, consider the content rule: it consists of two independent claims: (1) $(R \cap X) - Y = \emptyset$ and (2) $Y - S \cup X = \emptyset$. According to Kuipers's interpretation, (1) says that in all situations in which X is known to be true, Y is true; (2) adds that for all laws λ such that $S \vDash \lambda$ obtains, $X \vDash \lambda$ implies $Y \vDash \lambda$. Thus, at least for one R and S, the content rule keeps the two levels rigorously apart.[26] The refined rule, however, also relates structures, which, if *de*creasing in number, *in*crease logical strength, with structures, which, if *in*creasing in number, *in*crease logical strength. That is, it associates logical models of the conceptual level with structures of the application. In our monadic example, it associates *Q_i-instantiations with entire* *constituents*; and in our modal terminology, *it relates entire modal models* with elements of these models, namely *possible worlds*. If we keep the conceptual level and the level of application rigorously apart, the imperfection is readily repaired. First, we need *two* basic similarity notions; one for instantiating individuals, and one for the models. Then, it makes perfect sense to apply the instantial clause to the individuals instantiating the Q_i-predicates, and the explanatory clause to the constituents or models of the predicate language. The same holds, mutatis mutandis, for our modal proposal.

In the present subsection, we saw that a two-level reading of the correct data hypothesis solves the problems about the logical strength of the set R-instantiations, and the strongest accepted law S.

4.3.3.4 The truth of S

The correct data hypothesis consists of two parts. The first part is the assumption about the absence of failures in the data records. Although the hypothesis is idealistic from a practical point of view, from the theoretical standpoint it is acceptable. The second part of the hypothesis has an entirely different character. It assumes that all the logical models of $S(t)$ are models of T or, to put it differently, it assumes that the strongest accepted law S is *physically true*.[27] This assumption is too strong for two reasons. The first reason is a practical one; there is no S of which we know that it is factually true, and therefore the rule is inapplicable. As Hume has taught us, deducing a true general law from a restricted set of instances is impossible, and this lesson is almost generally accepted. Furthermore, the assumption of a true general law contrasts rather sharply with the rest of Popper's fallibilism. According to Popper, scientists will never know whether a law is true; they can only tentatively prefer one theory to another.

There is also a theoretical argument against the assumption of being able to formulate a series of true laws of increasing strength; it trivializes the epistemic problem of approach-to-the-truth. Suppose that, in the light of an increasing number of experiments, we could formulate stronger and stronger true empirical laws. Equipped with such a reliable inductive method, we did not need a methodological rule to approach the truth. Let $\varepsilon_1, ..., \varepsilon_n$ describe the evidence, and let $Cn(S) \supseteq \{\varepsilon_1, ..., \varepsilon_n\}$. If we assume the truth of the S, then S is definitely closer to τ than $\varepsilon_1 \wedge ... \wedge \varepsilon_n$. Consequently, the assumption we can formulate a series of physically true S_i of increasing logical strength implies that we can formulate laws that are certainly closer to the truth than the conjunctions of the observations on which the law is based. The need for a methodological rule based on truthlikeness evaporates, since the increase of the experimental material yields stronger true laws, and the strongest one is closest to the truth.

Having a reliable inductive method at our disposal, we can approach the truth without the success rule, but the inverse of this claim does not obtain. Approach to the truth remains uncertain when the success rule lacks the assumption of *S*'s physical truth. Thus, the functionality of the success rule is due to the truth of *S*. A more realistic approach uses posterior probabilities of laws and theories. It does not base the rule on the truth of *S* but on the probability of *S* given the evidence *R*; however, such a rule would cease to be functional.

To summarize, whereas the first part of the correct data hypothesis is rather innocent and acceptable, the second part is not. The physical truth of the strongest accepted law is practically and theoretically too strong. Only a reliable inductive method guarantees the functionality of the rule by ensuring that *T* implies *S*. More realistic answers to the epistemic problem of approach-to-the-truth incorporate posterior probabilities of the well-corroborated laws.

4.3.3.5 The Convexity Assumption

In subsection 4.3.2.3 we encountered the most important function of the convexity assumption in the refined proposal; we saw it guarantees the functionality of the rule. It is the basis of the claim that closer to the truth implies more empirical success. Moreover, the convexity assumption conceals most inconsistencies between the content and likeness rule. It is unclear, however, whether these advantages counterbalance the disadvantages. The assumption has been criticized, and, indeed, it is acceptable only as a first approximation. Perhaps, it was the convexity assumption that fostered the reluctance to give up classical mechanics seen at the start of the twentieth century. Today, however, it is unrealistic to assume that the truth must be convex, as one of the best physical theories is famous for its discontinuous behaviour. Originating from Planck's quantification of the energy levels of an oscillator, quantum mechanics derives its name from the denial of the convexity of nature. Consequently, a mature approach-to-the-truth definition cannot a priori assume the convexity of the truth. Moreover, Kuipers's comparative refined definition is better of without it. If we discard the convexity assumption, Kuipers's likeness proposal regarding finite propositional languages is similar to that of Brink and Heidema.

What would the result be of dropping the convexity assumption altogether? First, the essential difference between the structuralist content and likeness rules would become more apparent. Second, the likeness rule would cease to be functional for approaching the truth. This functionality, however, is conditional on the truth of the strongest accepted law *S*. Since this assumption must certainly be given up, it remains to be seen whether the costs of the convexity assumption counterbalance its profits.

4.3.3.6 Quantitative Problems

I end my comments and observations with two remarks about the structuralist quantitative rules of theory-choice. In the first place, at the end of Chapter 3, I showed that the *naive*, and *refined* definition have a *content* and *likeness* character, respectively. This corresponds with the proposals of the other investigators. In contrast to the refined quantitative proposals of Niiniluoto, and of Tichý and Oddie, the structuralist *refined quantitative* approach displays a *content* character.[28] In this approach, the negation of the truth is regarded as the worst possible theory. Moreover, both quantitative definitions suffer from the child's-play objection. That is, weak false theories are less verisimilar than strong ones. As the quantitative success rules are grafted onto the definitions, they exhibit analogous behaviour. The following table shows the content and likeness character of the structuralist rules and definitions.

Table 1. Structuralist Rules and Definitions

	Comparative		Quantitative	
	Naive	Refined	Naive	Refined
Definition	*Content*	*Likeness*	*Content*	*Content*
Rule	*Content*	*Likeness*	*Content*	*Content*

The second problem concerns the quantitative strategy and is more serious. According to Kuipers, a successful approach-to-the-truth explication must be functional in approaching the truth. That is, closer to the truth must, in some plausible way entail success predominance. His comparative proposals fulfill this constraint. Regarding the naive situation, the implication is straightforward. If we presuppose the truth of S, $Y <^{\Delta}_{T} X$ logically implies $Y \leq^{\Delta}_{R/S} X$ (prop. 4.1, p. 133). The refined strategy, however, causes more troubles; but they can be overcome. If S is true, and X, Y, T, and S are convex, then $Y \leq^{Kr}_{T} X$ generally implies $Y <^{Kr}_{R/S} X$ (prop. 4.3, p.135)

More persistent problems emerge with respect to the quantitative proposals. Neither the naive nor the refined quantitative definition, implies success predominance. First, naive quantitative closer to the truth, $Y <^{D}_{T} X$, does not imply naive quantitative success dominance, $Y \leq^{D}_{R/S} X$ (prop. 4.2, p.134). Second, although proposition 4.7 (p.138) shows that $Y \leq^{Dn}_{T} X$ increases the expectation of instantially success, we failed to formulate a suitable proposition for the explanatory part. Despite thorough research, the failure of which we shall spare the reader, the prospect of finding an acceptable relation between the refined quantitative definition and success predominance remains slim. Mainly, because the quantitative truthlikeness definition does not put constraints on $M_p - T$, leaving $\{S|\ T \subseteq S \subseteq M_p, |S| = s\}$ unspecified. The conclusion reads, presumably, that Kuipers quantitative definitions fail to be functional for approaching the truth.

4.3.4. The Advantages of the Modal Perspective

I presented my modal paraphrase of the structuralist approach in Chapters 2–3, and in this section, I discuss the advantages of this modal paraphrase of Kuipers's rules of theory-choice. In the first place, this paraphrase solves the question about the dynamic behaviour of $R(t)$ and $S(t)$. According to the structuralist paraphrase, the increase of R and the decrease of S both represent the increase of their logical strength. The modal formulation solves this problem since it carries through a stratified representation of R and S.

Let $\mathscr{L}_{S5}$ be the modal extension of $\mathscr{L}$, voc($\mathscr{L}$) = voc($\mathscr{L}_{S5}$), and let R be a subset of possible worlds (or non-modal constituents) representing the evidence accumulated thus far. If R is the *disjunction* of those worlds, then the increase of this disjunction would imply decrease rather than increase of the logical strength of the evidence. Thus, if

$$c_i \equiv (\pm)p_1 \wedge \ldots \wedge (\pm)p_n \in \text{Const}(\mathscr{L}) \text{ and } R = \vee_i\, c_i,$$

adding new c_j to R would decrease the logical strength of R (see subsection 4.3.3.3). If, however, R is a conjunction of constituents, R is the contradiction, if $R = \wedge_i\, c_i$, $1 < i < r$, then $R \equiv \bot$. The modal representation solves this problem, since the elements of R are *possible worlds*, that is, *elements* of the complete modal models, and S consists of those *complete modal models*:

$$R = \{w_1, \ldots, w_k\} \qquad (= \text{“}w_1 \text{ and } w_2 \text{ and, \ldots, and } w_k\text{”})$$
$$S = \{\mathfrak{M}_1, \ldots, \mathfrak{M}_m\} \qquad (= \text{“}\mathfrak{M}_1 \text{ or } \mathfrak{M}_2 \text{ or, \ldots, or, } \mathfrak{M}_m\text{”})$$

The syntactic representation of the evidence, then, becomes:

$$R := \wedge_i \Diamond c_i, \text{ and}$$
$$S := \vee_j\, C_j$$

with $c_i \in \text{Cnst}(\mathscr{L})$, $C_j \in \text{Cnst}(\mathscr{L}_{S5})$, ($i$ ranges over the “worlds we encountered” in experiments, and j ranges over the modal models that S does not exclude). Since R is a *conjunction* of physically possible worlds, and S is a *disjunction* of modal structures; increase of the first and decrease of the second both imply the increase of logical strength.

In the second place, the modal representation shows that the suggested parallel between an integral being “squeezed in” by a lower sum and an upper sum, and the truth being “squeezed in” between R and S, is misleading. A modal paraphrase of increasing evidence illustrates this fact; since then the correct data hypothesis assumes: $\{\mathfrak{W}\} \subseteq \mathrm{Mod}(S') \subseteq \mathrm{Mod}(S)$ and $\mathrm{Wd}(R) \subseteq \mathrm{Wd}(R') \subseteq \mathrm{Wd}(T)$ (cf. (10) and (11), p. 144); and the claims clearly concern different levels of the definition. New S's and R's *both strengthen* the evidence and T (= Th($\mathfrak{W}$)) is the complete truth.

Finally, the modal strategy explains the difficulties with the quantitative definitions. It discloses why the quantitative definitions do not imply a higher expectation of success predominance. The first problem is that the general structuralist approach neglects the structure of the set of models. For example, with respect to propositional languages, the set of all models of the language has the structure of a Boolean algebra. If Y is refined quantitatively closer to the truth T than X, nothing can be said about arbitrary sets S containing T; the rule discards the Boolean structure of T^c. Additionally, the quantitative structuralist approach combines distances of a different character. If we take our modal paraphrase of the structuralist approach-to-the-truth proposals seriously, it follows that the refined quantitative definition adds distances between models to distances between possible worlds, which are *elements* of those models. I shall not develop this modal perspective on Kuipers's refined quantitative proposal any further in the present study. In Chapter 6, however, I shall present my refined verisimilitude proposal, which provides a different approach to refined quantitative verisimilitude.

4.3.5. Conclusions

The structuralist proposal aims at a rule of theory-choice that is *functional* for approaching the truth. The comparative strategy succeeds in producing a functional *content* and *likeness* rule. Due to the physical truth of the strongest accepted law S, the first is functional for approach-to-the-truth. For convex theories and laws, the likeness rule also becomes functional for approaching the truth. This convexity assumption, however, unnecessarily renders many pairs of theories uncomparable and covers up the likeness character of the comparative refined rule. As to the quantitative strategy, neither the naive nor the refined definitions succeed in connecting approach-to-the-truth with the increase of expectation of success dominance.

Now that I have finished my presentation of Kuipers's approach-to-the-truth proposals, let me summarize the findings regarding the structuralist enterprise. In the second chapter, I made a distinction between the *conceptual* level and the level of *application* in Kuipers's one-tiered combination of received view and structuralism. The first level is concerned with the extension of the set of intended applications, and the second with the logical strength of the set theoretical predicate. I equated the strength of a theory with that of the *logical* strength of the predicate, as the representation of both aspects on one level turned out to yield paradoxical results. The one layer-representation covers up the important difference between the instantiations of a law and its (modal) models; this lead to difficulties concerning logical strength.

Kuipers started his "investigation of verisimilitude from the conviction (in the line of the structuralist approach) that empirical theories are not, and should not be complete ... ."[29] My inquiry showed, however, that a modal representation in which the truth is complete, provides an adequate representation of Kuipers's intuitions. The one-layered structuralist representation conceals the essential difference between the content and likeness definition, unnecessarily prompting a convexity assumption. Parallel to the innocent assumption of error-free data, $T \models R$, it also assumes that the strongest accepted law S is true, $T \models S$, which is too strong from a theoretical and practical point of view. Finally, regarding the (refined) quantitative case, the heterogeneous mixture of structuralist and logistic theory representation gives rise to the failure of relating the rule of success to the definition. Here, distances between instantiations of laws are added to distances between logical models of laws. The one-layered structuralist formulation would benefit greatly from incorporating the underlying modal intuitions; this requires a full-blooded modal representation.

4.4. NIINILUOTO'S ESTIMATED TRUTHLIKENESS

Niiniluoto's answer to the epistemic problem of truthlikeness is presented in this section. First Niiniluoto's ideas about *estimated truthlikeness* are introduced and I discuss some of its consequences. In the second part I shall sketch an odd consequence of the estimated truthlikeness, and finally I shall formulate an alternative.

4.4.1. The Definition and its Properties

In Chapter 3, we encountered Niiniluoto's general framework. This framework comprises a problem set $\mathbf{B} := \{h_i | \, i \in I\}$, and a distance function on $D(\mathbf{B})$, the set all partial answers. The elements of $\mathbf{B}$ are mutually exclusive and jointly exhaustive. Examples of $\mathbf{B}$ are constituents of a formal language, or possible values of some parameter.[30] Within this setting the question arises: how can we choose between theories regarding their truthlikeness, if we are ignorant of the truth?

I shall present Niiniluoto's answer to the epistemic problem by way of an example.[31] Consider a game in which a participant receives a sum of money, forecasts the outcome of a die throw, and then throws a die. The participant must pay back the amount of money, which is equal to the difference between the number on top and his or her guess. Suppose that the die is fair, what is the most reasonable bet on g? The answer is simple. If $\mathbf{X}$ is the random variable with possible outcomes 1, ... 6, the participant must minimize his or her expected loss

$$\mathscr{E}[|X-g|] :=_{def} \sum_{i=1}^{6} P(x_i)|x_i - g|$$

The participants answer must be as close as possible to the expected value of **X**. The rational guesses are $g = 3$, or $g = 4$.

Niiniluoto's rule of theory-choice is similar to the betting rule of the preceding example. Scientists gamble with theories, and the expected loss is the distance between their choices g and the truth. Consequently, according to Niiniluoto, the theory with the largest expected verisimilitude is to be preferred. Let P be an epistemic regular probability measure on a problem set and let evidence e fix the posterior probabilities of the complete answers. Let h_i, $i \in I$, be the elements of a problem set, and let g be a disjunction of h_i's, which represent a partial answer to the problem. Further, let $\mathrm{Tr}(g, h_i)$ represent truthlikeness measure on $D(\mathbf{B})\times\mathbf{B}$. Then, Niiniluoto defines "the *expected value* of the unknown degree $\mathrm{Tr}(g, h_i)$ on the basis of evidence e, relative to the probability measure P" by

$$\mathrm{ver}(g/e) :=_{def} \Sigma_{i \in I} P(h_i/e)\mathrm{Tr}(g, h_i)$$

According to Niiniluoto "ver(g/e) can be taken to be the *estimated degree of truthlikeness* of g on basis of e."[32] Mathematically, $\mathrm{Tr}(g, h_i)$ may be considered to be a *random variable* with domain **B** and codomain $\mathbb{R}$. Thus, analogous to the preceding example, ver(g/e) may be paraphrased by the mathematical expectation of $\mathrm{Tr}(g, h_i)$, $\mathscr{E}[\mathrm{Tr}(g, h_i)]$. Niiniluoto bases his "solution to the epistemic problem of truthlikeness" on this mathematical expectation, ver(g/e).[33]

g_1 is more truthlike on evidence e than g_2 iff
$\mathrm{ver}(g_1/e) > \mathrm{ver}(g_2/e)$

If the complete hypotheses are constituents of a monadic predicate language, Niiniluoto chooses Hintikka's inductive system to determine the probability distribution. In contrast to Carnap's continuum of inductive systems, Hintikka's λ-α system copes with infinite domains by using Bayes's formula, basing the prior probabilities and the likelihoods on Carnap's λ-continuum. Regarding more complex cognitive problems Niiniluoto also admits other methods of fixing probability distributions such as personalistic Baysian probabilities.[34] Probability measures on theories, however, still remain the object of serious philosophical debate. For example, Tichý and Oddie approve the general expected truthlikeness strategy, but disagree with Niiniluoto about the use of Hintikka's inductive systems.

To explain the ver-function, I will first summarize the Bayesian updating process of Hintikka's inductive system. Consider a finite monadic predicate language $\mathscr{L}$; and recall that a constituent of $\mathscr{L}$ is a proposition that describes for any

Carnapian Q-predicates whether it is instantiated or not. Then, at the outset, all constituents receive a prior probability based on *width* of the constituent, i.e. the number of filled Q-predicates. Next, the first incoming individual falsifies half of the constituents, and the λ-α system redistributes the probability mass among the remaining constituents. The corroborated constituents with 'small width' receive a relatively high probability, whereas the probabilities of the constituents with large width drop. Again, when new evidence fills an empty Q-predicate, falsifying some constituents, the probability mass is similarly redistributed among the remaining constituents. Suppose that, after some time, all new individuals instantiate non-empty Q-predicates. Then, the probability mass concentrates on the constituents close to the constituent that does not have empty Q_i-predicates and the a posteriori probability of the wider constituents receive increasingly lower probabilities. Finally, in Hintikka's λ-α system, the probability of the proposition is determined by Bayes's formula, which uses the posterior probability, and likelihoods of the constituents.

The updating process elapses asymmetrically. Instantiations of the Q_i-predicates will falsify the narrow constituents quicker than broad constituents, which will contribute to the updating process for a considerably longer period. Consequently, the width of the true constituent contributes to speed of the updating process. Note that the Clifford measure fits well to the updating method used in Hintikka's λ-α system. Wider (confirmed) constituents receive a lower probability than narrow ones, and the Clifford measure reinforces this result of Hintikka's inductive method.

Let us summarize Niiniluoto's answer to the epistemic problem. Niiniluoto uses the mathematical expectation of the random variable Tr(g, **H**) to estimate the truthlikeness of g. He uses Hintikka's Bayesian conditionalization to determine the posterior probabilities. The probability mass concentrates around the true constituent; and the answer with the highest expected truthlikeness has the best chances of being closest to the truth.

Now that I have summarized the rule of estimated truthlikeness, let us turn to some properties of the ver-function as put forward by Niiniluoto. If Tr(g, h_i) := $1-\Delta_{ms}^{\gamma\gamma'}$ ($\Delta_{ms}^{\gamma\gamma'}$ is the *min-sum* measure and equals $\gamma\Delta_{\min}(h_i, g) + \gamma'\Delta_{\text{sum}}(h_i, g)$), then ver($g/e$) equals

$$(12) \qquad \text{ver}(g/e) = 1 - \sum_{i \in I} P(h_i/e) \left(\gamma\Delta_{\min}(h_i, g) + \gamma' \frac{\sum_{j \in \mathbf{I}_g} \Delta_{ij}}{(|I|\,\text{av}(i, \mathrm{B}))} \right)$$

since $\sum_{i \in I} P(h_i/e) = 1$. I shall consider two special cases: 1. the case in which g is a *complete answer* h_i; and 2. the case in which we are, in Niiniluoto's words, *completely ignorant*, i.e. $P(h_i/e) = |I|^{-1}$ for all $i \in I$, and av(i, **B**) = ½. First, let us assume that g is a complete answer; then $\Delta_{\min}(h_i, g)$ and $\sum_{i \in I} \Delta_{ij}$ reduce to Δ_{ij} and ver(g/ e) boils down to

$$(13) \qquad \text{ver}(h_j/e) = 1 - \sum_{i \in I} P(h_i/e)\left(\gamma + \frac{\gamma'}{(|I|\text{av}(i,\mathrm{B}))}\right)\Delta_{ij}$$

Second, consider the situation of complete ignorance where the problem set is balanced, i.e. av(i, **B**) = ½, $i \in I$, and the, in this case priori, probability distribution given e, equals $P(h_i/\mathrm{e}) = |I|^{-1}$, $i \in I$. Kuipers calls this situation an *a priori system*. Under these circumstances, ver(g/e) equals:

$$(14) \qquad \text{ver}(g/e) = 1 - \frac{\gamma}{|I|}\sum_{i \notin \mathbf{I}_g} \Delta_{\min}(h_i, g) - \frac{\gamma'|\mathbf{I}_g|}{|I|}$$

The derivation of the latter formula is elementary and can be found in Niiniluoto (1987) page 272. Note that in these circumstances, if g is a tautology, ver(g/e) = $1 - \gamma'$. Further, if g is a complete hypothesis, the latter formula reduces to

$$(15) \qquad \text{ver}(h_j/e) = 1 - \frac{\gamma}{2} - \frac{\gamma'}{|I|}$$

as $|\mathbf{I}_g| = 1$, and $\sum_{i \notin \mathbf{I}g}\Delta_{ij}|I|^{-1} = \sum_{i \in I}\Delta_{ij}|I|^{-1} = \text{av}(i, \mathbf{B})$. I want to stress that the three situations of complete ignorance, or, of an a priori system, concern extreme borderline cases.

The preceding formulae imply some important characteristics of Niiniluoto's epistemic rule. The first one is that the *posterior probability*, or confirmation, of a hypothesis *differs* from its *estimated verisimilitude*.[35] I illustrate this difference in three ways. First, equation (13) implies that new evidence may *increase* the probability of the theory g whereas it *decreases* its estimated truthlikeness:

$$\text{ver}(g/e) > \text{ver}(g/e \wedge e') \text{ and } P(g|e) < P(g|e \wedge e')$$

I give an example in Table 2. The complete answer h_1 is close to h_2, and h_2 is at large distance form h_3; the same obtains for h_1 and h_3. Further, old evidence e makes h_1 improbable, but provides h_1 with a high degree of estimated truthlikeness, the likely h_2 is in the neighbourhood of h_1.

New evidence e' changes this situation. It falsifies h_2, which decreases the estimated truthlikeness of h_1 but increases its probability. Second, the ver-function allows that $\Delta_{ij} \approx 0$, ver($h_i|e$) ≈ 1 and $P(h_i|e) = 0$, it is not truth-value dependent.[36] If, in the preceding example, h_1 was falsified by e, and $P(h_2|e)$ was 0.875, then ver($h_1|e$) would be equal to $1-0{,}121\gamma-0{,}048\gamma'$. Third, among the true hypotheses the stronger g_1 has a higher estimated degree of truthlikeness, whereas the weaker

g_2 is more probable.[37] More explicitly, let h^* be the truth: $P(h^*|e) = 1$, and e has falsified all other h_i, then

$$\text{If } h^* \vdash g_1 \vdash g_2 \text{ and } g_2 \nvdash g_1, \text{ then } \mathrm{ver}(g_1|e) > \mathrm{ver}(g_2|e).$$

This claim is entailed by equation (12). Since $P(h^*|e) = 1$, $P(h_i/e) = 0$ for all $h_i \neq h^*$; then for true g, ver(g/e) reduces to $1-\gamma'\Delta_{\mathrm{sum}}(h^*, g)$, and the larger the true g, the smaller its expected truthlikeness. Consequently, the tautology is the worst of the true consequences. These three examples clearly show the difference between expected verisimilitude and probability.

Table 2.

	h_1	h_2	h_3
Δ	$\Delta_{12} = 0{,}01$	$\Delta_{23} = 0{,}89$	$\Delta_{31} = 0{,}9$
$P(h_i\|e)$	⅛	¾	⅛
ver(h_i/e)	$1-0{,}12\gamma-0{,}05\gamma'$	$1-0{,}11\gamma-0{,}04\gamma'$	$1-0{,}78\gamma-0{,}58\gamma'$
$P(h_i\|e \wedge e')$	¼	0	¾
ver($h_i/e \wedge e$)	$1-0{,}66\gamma-0{,}26\gamma'$	$1-0{,}67\gamma-0{,}25\gamma'$	$1-0{,}23\gamma-0{,}17\gamma'$

A second important property of Niiniluoto's answer to the epistemic problem is the *preference of the tautology* $\top$ *to any complete answer in the case of complete ignorance* if (roughly) $2\gamma' < \gamma$.[38] It follows from the equation (14) that according to the a priori system, ver($\top$, e) = $1 - \gamma'$ ($I_g = I$ and for all h_i the minimum distance to $\top$ is zero). Equation (15) gives the estimated truthlikeness of the complete answers in the case of complete ignorance. Then,

$$\mathrm{ver}(\top, e) > \mathrm{ver}(h_i, e) \text{ iff } \gamma' < \tfrac{1}{2}\gamma + \frac{\gamma'}{|I|}$$

and since $|I|$ is generally large, this applies if, roughly, $2\gamma' < \gamma$.

The third important property of Niiniluoto's estimated truthlikeness is that *in the limit the estimated truthlikeness of g approaches its actual truthlikeness*, in other words: Niiniluoto's answer is effective in approaching the truth. This result follows again from equation (12). If the amount of collected evidence increases without a limit, there will be a point in time after which a large number of h_i are falsified, and all new incoming evidence lowers the probability of all $h_i \neq h^*$. In these circumstances, it follows from the first formula that the estimated truthlikeness approaches the actual truthlikeness. Niiniluoto gives the following exact formulation:[39]

PROPOSITION 4.10: If $P(h_j/e_n) \to 1$, $n \to \infty$, for some $j \in J$; and $P(h_i, e_n) \to 0$, $n \to \infty$, for $i \neq j$, $i \in I$; then ver(g/e_n) $\to$ Tr(g, h_j), $n \to \infty$.

The final important feature of Niiniluoto's proposal that I want to mention is its *reversibility.*[40] It may occur that new evidence e' reverses the original preference order $\text{ver}(g/e) < \text{ver}(f/e)$ into $\text{ver}(g/e \wedge e') > \text{ver}(f/e \wedge e')$. We have already encountered such an inversion of preference in Table 2 where $\text{ver}(h_3/e) < \text{ver}(h_2/e)$ and $\text{ver}(h_3/e \wedge e') > \text{ver}(h_2/e \wedge e')$. Consequently, $\text{ver}(g/e) > \text{ver}(f/e)$ does not exclude $\text{Tr}(g, h^*) < \text{Tr}(f, h^*)$, and the ver-rule is not irreversible in the technical sense of the word; however, proposition 4.10 shows that someone who follows the rule effectively approaches the truth.

4.4.2. Expected Distance versus Distance to Mean

In this subsection, I comment on the estimated truthlikeness rule. First, I shall make a terminological remark, and then I shall raise the main point of my critique. To begin with, I want to stress that Niiniluoto uses x^* in two incompatible ways.[41]

"Let $x^* \in \mathbb{R}$ be the unknown value of a quantity and let x_i, $i \in I$, be the possible values that x^* could have. (...) Then the expected value of x^* on the basis of e and P is $\sum_{i \in I} P(x_i|e)\, x_i$ (*Niiniluoto* (1987) p. 268)"

Strictly speaking, the meaning of x^* before the ellipsis is incompatible with the meaning of x^* after the ellipsis. The "expected value of x^*" of the second part implies that x^* is a *random variable*. Then, one commits a category mistake by saying "Let $x^* \in \mathbb{R}$ be the unknown value of a quantity." If the latter were true, however, it is meaningless to consider the *mathematical expectation* of x^*. The introduction of an extra (random) variable solves the problem. The situation is more adequately described as follows.

"Let x^* be the unknown actual value of a quantity. Furthermore, let $\mathbf{X}$ be a random variable with the possible values x_i of x^* as its range. If $P(\mathbf{X} = x_i|e)$ is the rational degree of belief that $x^* = x_i$, we can choose the mathematical expectation of $\mathbf{X}$, $\mathscr{E}(\mathbf{X}) := \sum_{i \in I} P(\mathbf{X} = x_i|e)\, x_i$, as our estimate of the unknown value of x^*."[42]

The ver function has the same minor imperfection as the truthlikeness definition, and it is removed equally easily. Again, $\text{Tr}(g, h^*)$ is called "the unknown degree of truthlikeness ... of g" but also "the possible values of $\text{Tr}(g, h^*)$ are $\text{Tr}(g, h_i)$, $i \in I$." Similarly, if $\text{Tr}(g, h^*)$ is a fixed similarity, it only has *one fixed value*. Consequently, the situation is more adequately described if we consider $\text{Tr}(g, \mathbf{H})$ to be a *random variable* that associates a real-valued distance to every $h_i \in \mathbf{B}$ and the specfied g. Let $P(\mathbf{H} = h_i|e)$ be a rational degree of belief on basis of e that h_i is the actual value ($h^* = h_i$). Then, we may choose the *expected value* of $\text{Tr}(g, \mathbf{H})$ as estimate for the unknown $\text{Tr}(g, h^*)$ (if h_i are numerical values, and g is a set of those values):

$$\text{ver}(g/e) = \mathscr{E}\,[\text{Tr}(g, \mathbf{H})] = \sum_{i \in I} P(\mathbf{H} = h_i|e)\text{Tr}(g,h_i)$$

In sum, Niiniluoto chooses as an estimate of the unknown value Tr(g, h^*), the (mathematical) expectation of the random variable Tr(g, **H**), with range Tr(g, h_i), $i \in I$.

What is the reason, one might ask, for contributing space and time to such an easily removable imperfection of the description of the rule? The answer is two-fold. In the first place, it is incorrect to consider the *expected value* of x^* when $x^* \in \mathbb{R}$ is a fixed value. The main reason, however, is to warn the unwary reader that ver(g/e) is not the estimate of the similarity between g and the truth h^* in the same vein as the statistical mean estimates the actual mean of a population. Calling the expectation of Tr(g, **H**), ver(g/e), "the *expected value* of the unknown degree Tr(g, h^*)," does not particularly prevent such misunderstanding.[43] Even more confusingly, the expectation of $\Delta(g, \mathbf{H})$ is called "the *expected distance* of *g from the truth* h^*."[44] The former expression is a somewhat inadequate description of the fact that Niiniluoto *chooses to use the mathematical expectation of* $\Delta(g, \mathbf{H})$ as *an estimate for* $\Delta(g, h^*)$.

Although ver(g/e) seems to be a reasonable choice if the problem consists of qualitative entities — e.g. constituents — it may lead to unexpected results as far as quantitative problem spaces are concerned. For instance, if the theories are about the isotopes of an element, **H** already is a random variable; and Tr(g, **H**) is a compound random variable. An interesting question reads whether ver(g/e) varies with the similarity between the expectation of **H** and g, if **H** is a quantitative random variable. More exactly the question reads, does

$$\mathscr{E}[\mathrm{Tr}(g,\mathbf{H})] > \mathscr{E}[\mathrm{Tr}(g', \mathbf{H})] \text{ imply } \Delta(\mathscr{E}[\mathbf{H}], g) < \Delta(\mathscr{E}[\mathbf{H}], g')$$

if **H** is a stochast. In other words, will the results be the same if we first calculate all distances between h_i and g and then determine the expectation of those distances; or to reverse this order, if we first calculate the expectation of the h_i and then measure the distance between this expectation and g. As we shall see in the following paragraphs, the answer is negative. Hence, our critique of choosing the expectation of Tr(g, **H**) as the estimated degree of truthlikeness reads:

> For quantitative problem spaces, the mathematical expectation of the random variable Tr(g, **H**), may deviate from the mathematical expectation of **H**.

The two following examples show why this difference is problematic. First, let us consider a gambling game with one fair die. After casting the die, we receive $2n$ units of value, if for n, the number of spots on top, obtains $n \leq 3$; and we receive $n + 3$ units of value if $n > 3$. We are rational if we accept to pay six or fewer units of value to participate in the game. In terms of the cognitive problem, the complete answers h_n are the complete outcomes of the problem space, and $\mathscr{E}[\mathbf{H}] := 3$. Some arithmetic based on equation (13) shows us that, according to the ver-function, the best theory in this cognitive problem is the same as the theory closest to the expectation of **H**: for all h_i: ver(h_i,⊤) < ver(h_3,⊤).

If, however, the bank pays us 10^n units of value under the same conditions, it is rational to claim that the game is worth $^1/_6\Sigma_{i=1}^{6}10^i = 185185$ units of value since this is the expectation of this **H**′.[45] The hypothesis closest to the expected outcome is $h_5 = 5$ such that **H**′ = 105; however, the outcome with the highest *expected truthlikeness* is $h_3 = 3$, and **H**′ = 10^3.[46] Thus, in the a priori situation in which there is no evidence whatsoever, the ver rule and the mathematical expectation of the random variable **H** provide *different advice*. How must this difference of estimation be interpreted? Why should we pay more than a hundred thousand units of value to participate in a game in which the outcome with the highest degree of truthlikeness is only a thousand units of value?

Let us turn to our second example. It shows that the contrast between mean and estimated truthlikeness also creates difficulties for probabilistic hypotheses. Consider a large population of six kinds of balls: the 10, 10^2, ..., and 10^6-balls. Suppose we are unfamiliar with the fact that the number of all kinds of balls is the same; hence, we do not know that the mathematical mean of **H** is 185185. Consider the following cognitive problem: Which number on the balls is the closest to the mean of the population? We take many samples of n balls, and statistically we come to the posterior probability distribution of the hypothesis about the mean as conveyed in Table 3. As we have taken many samples, and the expectations of the sample distribution of means is equal to the population mean, statistically, we would correctly choose h_5 as the best guess for the mean of the population.

As to the ver rule regarding statistical hypotheses, it seems plausible to equate the degrees of beliefs with relative frequencies. Then, the estimated degree of truthlikeness of h_4, is higher than that of hypothesis h_5.[47] Thus, it is unclear whether we should choose the most probable outcome, or the outcome with the highest estimated degree of truthlikeness. We may conclude that, regarding statistical hypotheses, there are important differences between "closest to the mean" and "highest estimated degree of truthlikeness".

Table 3. The Posterior Probability Distributon of **H**

H	101	102	103	104	105	106
$P(h_i \approx \mathscr{E}\,[\mathbf{H}]\,|\,e)$:	$^1/_{21}$	$^2/_{21}$	$^3/_{21}$	$^5/_{21}$	$^6/_{21}$	$^4/_{21}$

In the preceding paragraphs we proved the next observation. Suppose $h_1, \ldots, h_n$ are the complete answers of quantitative cognitive problem **B**, and g and g' are two partial answers to the problem; let **H** be a random variable with the range $h_1, \ldots, h_n$; then

PROPOSITION 4.11: $\mathrm{ver}(g/e) < \mathrm{ver}(g'/e) \not\Leftrightarrow \Delta(\mathscr{E}[\mathbf{H}], g) \geq \Delta(\mathscr{E}[\mathbf{H}], g')$

For quantitative problem sets, it is worthwhile to examine the replacement of the expectation of Tr(g, **H**) by the distance between a partial answer and the mean, $\Delta^{\gamma\gamma'}_{ms}(\mathscr{E}[\mathbf{H}], g)$. This proposal certainly fares better as far as the difference with the expectation of **H** is concerned. Moreover, the alternative also fits well to Niiniluoto's use of Bayesian conditionalization.[48]

In sum, Niiniluoto chooses the expectation of the random variable Tr(g, **H**) as the estimation of the degree of truthlikeness. A reasonable adequacy condition for this estimation is that for quantitative cognitive problems the preference order according to the distance to the mean of **H**, $\Delta(\mathscr{E}[\mathbf{H}], g)$ equals the preference order according to the expected truthlikeness, if there is no evidence available. Regarding these quantitative problem sets, Niiniluoto's choice fails to meet this constraint, and the implication of this failure needs further investigation.

4.5. GENERAL COMPARISON OF THE RULES

In Sections 4.3–4.4, we encountered Kuipers's rule of theory-choice, the rule of Niiniluoto and our local comments. In this section I want to compare both rules using the global characteristics mentioned in Section 4.1.

4.5.1. Congruence between Definition and Rule

As mentioned in Section 4.1, one would expect a correspondence between the order fixing elements of the *definition*, and the elements that settle the order according to the *rule*. To what extent do the rules come up to these expectations?

I have already pointed at the similarity between Kuipers's definitions and his rules. Within his content approach, true and false consequences and antecedences fix the similarity to the truth; and the same holds for the preference order of the rule. As to the refined approach it is similarity between models that establishes the order of the definition; and again, the same is true for the rule. In brief, within both structuralist strategies there is a reasonable correspondence between the order-fixing elements of the definition and those of the rules.[49]

If we choose the expectation of Tr(**H**,g) as the estimator of Tr(h^*,g), the relation between the definition and the rule becomes more opaque. As Bayesian conditionalization forms the foundation of the ver function, *consequences* of the truth establish its preference order. For the truthlikeness definition, the consequence relation plays merely a subsidiary role. It is the similarity between constituents that establishes the truthlikeness order. That $\neg p \vee \neg q$ does not have a true empirical consequence does not prevent it from being closer to the truth $p \wedge q$ than

$\neg p \wedge \neg q$, which has several true consequences. Of course, the rational degree of belief, or probability is not the only ingredient of the ver function. Niiniluoto describes ver(g,e) as "the *weighted average* of the possible values of Tr(g, h^*), where each of these values Tr(g, h_i) is weighted with the degree of belief $P(h_i, e)$ on e that h_i is true."[50] Rational degree of belief, however, is the only *empirical* ingredient of ver(g/e). The truthlikeness part is an analytical or *a priori* factor. In short, the elements that establish Niiniluoto's truthlikeness order differ from those that fix the preference ordering of the rule. Where *similarity between constituents* primarily establishes the theoretical truthlikeness of a theory, Niiniluoto bases the *estimated truthlikeness* of a theory on the true consequences of the confirmed complete answers.

Our conclusion must read, then, that regarding the congruence between the definition and the answer to the epistemic question, the structuralist approach shows more coherence than the quantitative truthlikeness approach.

4.5.2. Application of the Rule

Our second global criterion concerns the applicability of the rules. Regarding the application of the ver function, we must consider two different situations: first, rather restricted quantitative situations, and second, the more general contexts where sentences are to be compared. If the first obtains, e.g. point estimates, interval estimates, etc., the ver function resembles Bayesian conditionalization.[51] There is a restricted set of conceptual possibilities and a personalistic probability distribution. In these situations, the ver function has a practical value. If the cognitive problem concerns the choice between general propositions that represent scientific theories, the application of the ver rule is considerably more difficult.[52] Then, again, we must distinguish between the situations in which the probability measure has a frequency interpretation, and those in which only personalistic interpretations are available. Indeed, if monadic propositions paraphrase all complete theories, and, if an urn model represents the probabilities involved, then individuals belonging to some Q-predicate may paraphrase incoming evidence. In such an elementary problem situation, the ver function may estimate objectively the truthlikeness of a theory.

Unfortunately, scientific decision situations are much more complicated than this; in the first place, few scientific theories are axiomatizable in a formal language with manageable constituents. It is not realistic to assume that a scientist can consider the set of all conceptually possible alternative theories. Secondly, the events involved will have a unique character, which excludes a frequency interpretation of the probabilities. Moreover, there is no generally accepted theory of logical probability that handles the extremely complicated case of scientific theories. Consequently, Niiniluoto must fall back to subjective probabilities, and it remains to be seen whether this still guarantees objective decision procedures and scientific progress.[53] Obviously, Niiniluoto is well aware of this situation, his answer to these objections is that the ver rule is not to be considered as an actual rule of theory-choice. It is rather a *rational reconstruction* of how the scientific theory preferences come about.

As Niiniluoto claims that his answer to the epistemic question is not applicable to scientific practice, perhaps the content rule has more practical value. Unfortunately, Kuipers's content rule is not readily applicable either, as it is based on assumptions that are too strong. First, all theories, including the true one, must be convex. Secondly, we must have a definitely true law *S* at our disposal, and thirdly, the better theory must improve on the other theory in all respects.

Regarding the application of the rules, we must conclude that, with respect to linguistically formulated theories, both rules are difficult to apply; either the theory is based assumptions that are too strong, or it needs constituent representation.

4.5.3. The (ir)reversibility of the Answers to the Epistemic Problem

In Section 4.3, we saw that Kuipers's comparative content rules functionally approach the truth (equat. (8), p. 139, and corollary 4.4, p. 135). Furthermore I proved that the naive success rule is irreversible (proposition 4.8. p. 140). That is to say, if, according to the evidence collected up to now, the theory ψ is (strictly) preferred to φ, then no future evidence whatsoever can reverse that strict preference order. Even if all future evidence favours φ instead of ψ, the success rule will not (strictly) favour φ to ψ. The theories become uncomparable, as new evidence cannot delete the crucial experiments favouring ψ.

For these reasons, Kuipers calls his rule "nonfrustrating," or "functional for approaching the truth".[54] Kuipers's "constructive realism" and his approach-to-the-truth definition together explain the everlasting supremacy of the empirical success of theories that are closer to the truth. Alternative rules such as, for example, "choose the best problem solving theory" lack such explanation. Given this, Kuipers claims that he has successfully taken up Laudan's challenge.

It can be seen from Table 2, in the light of new evidence, Niiniluoto's ver-function may reverse a preference order. At the outset, when little evidence is available, the probability distribution over the constituents may be very different from the distribution after the bulk of the evidence has come in. This can influence the truthlikeness estimation to such an extent that the accumulation of evidence reverses the initial preference order (cf. the next example). Consequently, being more truthlike does not imply being favoured by the ver-rule. Nevertheless, obeying the rule of expected truthlikeness eventually implies "heading for the truth." As we have seen in subsection 4.4.1, in the limit the expected truthlikeness equals the actual truthlikeness (Prop 4.10, p. 154). In sum, in contrast to the success rules, the rule of expected truthlikeness is reversible.

In the next example I compare the behaviour of Kuipers's rule of success and Niiniluoto's rule of estimated truthlikeness. To that end, I shall use my two-level interpretation of the success rules to avoid their paradoxes of logical strength. Furthermore, there is the inclination to think that we should compare Kuipers's likeness rule with the estimated truthlikeness rule. As, however, the likeness success rule relates specific elements of *X*, *Y* and *R/S*, it requires a profound revision to avoid the paradoxes of logical strength. The most plausible adjustment distinguishes between an instantiation and an explanatory clause; and additionally it must define a similarity relation for constituents, and one for individuals instantiating the Q_i-predicates. As this proposal would change Kuipers's original approach substantially, it is more appropriate to use his content rule since the latter only requires that $Y-(X\cup S) = (R\cap X)-Y = \varnothing$. Moreover, both success rules fail to decide between theories *X* and *Y*, if there are crucial experiments in favour of *X* and *Y*; and it is this situation that most clearly illustrates the difference between the rule of success (p. 134) and the rule of estimated truthlikeness.

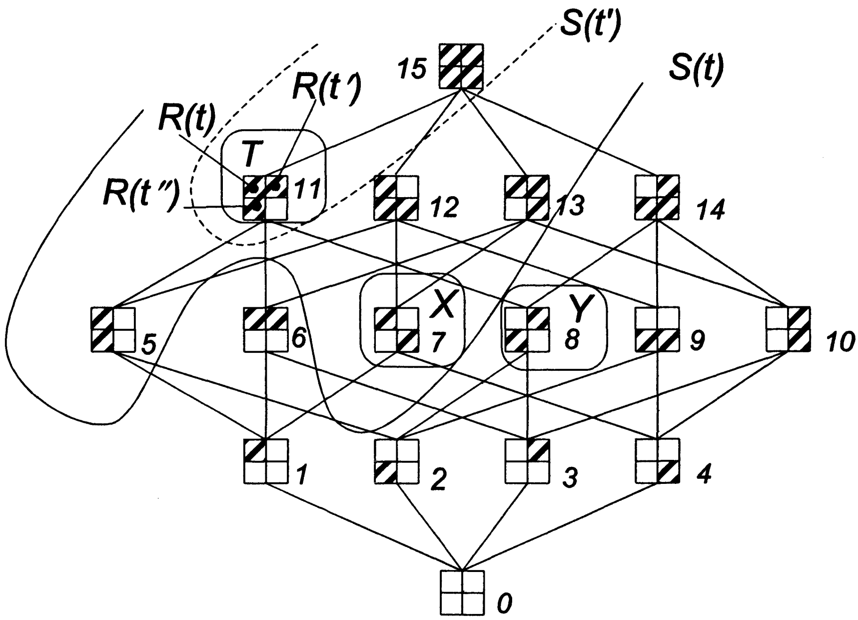


Fig. 9. Stratified representation of success rule

EXAMPLE: In this example I compare the behaviour of the content success rule and the rule of estimated truthlikeness. Consider the monadic predicate language

$\mathcal{L}[V(x), W(x)]$ given in the example on page 143 without names for individuals, and with vocabulary $\{V(x), W(x)\}$. As before, the ⊞-boxes in the diagram systematically represent the constituents of $\mathcal{L}$. In this example theories X, Y and T are all complete. They are equivalent to constituents C_7, C_8 and C_{11}, respectively:

$$X := \exists x Q_1(x) \wedge \exists x Q_4(x) \wedge \forall x[Q_1(x) \vee Q_4(x)]$$
$$Y := \exists x Q_2(x) \wedge \exists x Q_3(x) \wedge \forall x[Q_2(x) \vee Q_3(x)]$$
$$T := \exists x Q_1(x) \wedge \exists x Q_2(x) \wedge \exists x Q_3(x) \wedge \forall x[Q_1(x) \vee Q_2(x) \vee Q_3(x)]$$

First, suppose that until time t, $R(t)$ equals a set of individuals instantiating Q_1; and let the strongest accepted law $S(t)$ be the disjunction of constituents C_5, C_7, C_{11}, C_{12}, C_{13} and C_{15}. Set $R(t)$ and $S(t)$ can be represented by sentences of $\mathcal{L}[V, W]$;

$$R(t) := \exists x Q_1(x)$$
$$S(t) := (\exists x Q_1(x) \wedge \exists x Q_3(x) \wedge \forall x[Q_1(x) \vee Q_3(x)]) \vee C_7 \vee C_{11} \vee C_{12} \vee C_{13} \vee (\exists x Q_1(x) \wedge \ldots \wedge \exists x Q_4(x) \wedge \forall x[Q_1(x) \vee \ldots \vee Q_4(x)])$$

Then, both the success rule and the rule of estimated truthlikeness favour X to Y. Evidence $R(t)$ confirms constituents C_1, C_5, C_6, C_7, C_{11}, C_{12}, C_{13} and C_{15}, and the average of the Clifford-distances between X and these constituents is smaller than the average distance between these constituents and Y. Suppose, now, that at the second stage of the investigation besides Q_1-instantiations scientists also encounter Q_2-instantiations, which leads them to provisionally accept law $S(t')$. Then:

$$R(t') := \exists x Q_1(x) \wedge \exists x Q_2(x)$$
$$S(t') := \exists x Q_1(x) \wedge \exists x Q_2(x) \wedge \exists x Q_3(x) \wedge \forall x[Q_1(x) \vee Q_2(x) \vee Q_3(x)] \vee \vee (\exists x Q_1(x) \wedge \ldots \wedge \exists x Q_4(x) \wedge \forall x[Q_1(x) \vee \ldots \vee Q_4(x)]) \equiv C_{11} \vee C_{15}$$

Since $\exists x Q_1(x)$ corroborates X and falsifies Y and $\exists x Q_2(x)$ corroborates Y and falsifies X, X and Y are uncomparable according to the success rule. Not so for the estimated truthlikeness rule. At time t', only constituents C_6, C_{11}, C_{13}, and C_{15} contribute to the estimated truthlikeness of X and Y. From considerations of symmetry we see that they favour neither X nor Y; and Niiniluoto's rule treats X and Y on a par. Let us suppose that, eventually, the scientists also encounter Q_3-instantiations, and that they stay with $S(t')$ as the strongest accepted law:

$$R(t'') := \exists x Q_1(x) \wedge \exists x Q_2(x) \wedge \exists x Q_3(x)$$
$$S(t') := S(t'').$$

This means that only constituents C_{11}, and C_{15} contribute to the estimated truthlikeness of X and Y; and as they are closer to Y than to X, in this last situation the estimated truthlikeness rule favours Y to X.

Leaving out the stratified character, Figure 10 represents the preceding example using Venn diagrams. X, Y and T are sets of $\mathcal{L}[V(x),W(x)]$-constituents; in this example they are singletons. $R(t)$ designates instantiated Q-predicates and S is a complement of a disjunction of correctly excluded constituents. The shaded areas symbolize sets of empty Q-predicates. A "+" designates an instantiation of a Q_i-predicate, and a "×" the presence of constituent; the content of non-shaded areas is irrelevant. At time t, the success rule favours X to Y, since X has a realizable example + that falsifies Y; it is a crucial experiment that favours X. The rule of expected truthlikeness gives the same result. Note that constituent $\times \in Y-(X \cup T)$ is a virtual counterexample for Y.

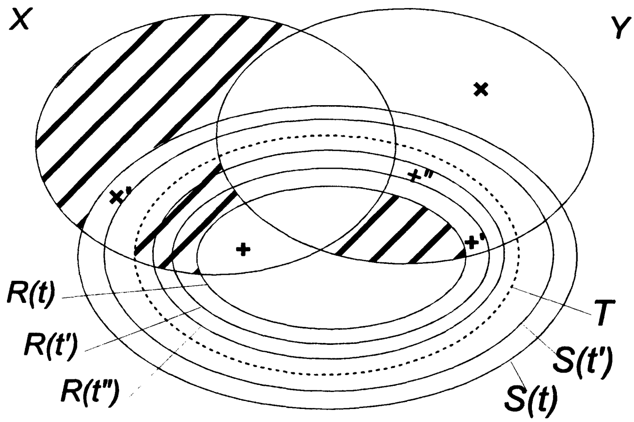


Fig. 10. Unstratified of updating process

At time t' however, X also has a realizable counterexample $+'$ that is an example of Y; $+'$ is a crucial counterexample favouring Y. Additionally X has a virtual counterexample $\times'$ that is also a virtual example of Y. This makes X and Y uncomparable according to the success rule. From considerations of symmetry we may conclude that the expected truthlikeness rule, considers the two theories equally truthlike. At the final stage, where also Q_3-instantiations +" have turned up, X and Y remain uncomparable according to the success rule, but the estimated truthlikeness rule will eventually favour Y to X. Note that if X, Y and T are sets of constituents of more than one constituent plus instantiations, then the unstratified representation of the updating process using Venn diagrams becomes rather confusing. The conclusion of our example reads that if there are crucial experiments against X and Y, then X and Y remain uncomparable as far the success rules are concerned, whereas the estimated truthlikeness rule may still establish a preference order. *End Example*

As Kuipers has proposed an irreversible rule and Niiniluoto a reversible one, the researchers apparently disagree about the adequacy conditions of the answer to the epistemic approach-to-the-truth question. Niiniluoto prefers a rule matching the actual decision processes, where new evidence may reverse the old preferences order. Kuipers chooses a rule serving a theoretical purpose: closer to the truth deductively implies increasing empirical success. These points of view may be less incompatible than they seem on first sight. From thermodynamics we learned that probabilistic behaviour on micro level may be reconcilable with functional or even deterministic phenomena on a macro level. Perhaps the probabilistic nature of the estimated truthlikeness relates to the determinism of the success rule as statistical thermodynamics stands to phenomenological thermodynamics. Although conjectural on the micro level, due to large quantities of evidence on a macro level, the truthlikeness rule may irreversibly explain the lasting success of theories.[55] As we can use Niiniluoto's ver function to compare more triads of theories, relative to Kuipers's rule, this function appears to operate on a micro level.

Finally, the preceding example shows that reversibility of the estimated truthlikeness is closely connected to quantitative methods, in particular probabilities and distances to the truth. It remains to be seen whether this is a necessary condition for a rule to be reversible. There seems to be no conclusive, a priori argument, blocking a combination of qualitative probabilities and a comparative approach-to-the-truth definition.

4.5.4. Complete Truth

All the answers to the epistemic problem considered in this chapter presuppose that the truth is complete. Although Kuipers seems to disagree about the incompleteness of the truth, his rules assume a dichotomy between the set of physically possible and the set of empirically impossible worlds. Consequently, in our modal paraphrase the truth is complete. Although Niiniluoto provides a definition for indeterminate languages, his answer to the epistemic problem is also based on the same assumption of completeness. In brief, all answers to the epistemic problem thus far assume a fixed truth-value for all contingent sentences in the language. This is quite remarkable. As sketched in Section 4.1, scientific researchers rarely assume that their conceptual framework is completely adequate. They do not know whether all theoretical notions in their language provide truth-values for all sentences containing these notions. After all, scientists have worked, even successfully, with notions such as "phlogiston", and "ether". Thus from this practical point of view, it is rather improbable that an arbitrary conjunction of two distinct conceptual frameworks establishes a determinate language, yielding a complete truth. Therefore, an answer to the epistemic problem also has to provide a preference order for an indeterminate conceptual framework. Neither the success rule nor the estimated degree of truthlikeness considers this standard situation.

It remains to be seen whether an answer to the epistemic problem plays a role in choosing the most appropriate language of investigation. On the one hand, the introduction of a new notion is sometimes clearly a step in the right direction, e.g. the introduction of the oxygen notion. On the other hand, to answer the comparative epistemic problem one has to choose between existing theories; thus, such an answer does need to introduce new notions, it indicates which of the theories may be closer to the truth within the given conceptual framework. Whether, and if so, how, approach to the truth can influence the choice of the most adequate conceptual framework is a question that I shall leave for future investigations.

Finally, I want to formulate my definite judgement of Kuipers's incomplete truth assumption. As a start, it is almost generally accepted that Kuipers's modal intuitions about empirical theory representation are fruitful. His dichotomization of the universe of logical possibilities into physically possible and impossible worlds makes sense. There is, however, considerably less sympathy for the stress Kuipers places on the distinction between the instantiations of a theory and its laws.[56] It is generally accepted that the relation between a theory and its instantiations is the same as that between the theory and its laws. If we take together the background information and the initial conditions, according to Kuipers's proposals the instantiations *and* the laws *are both entailed* by the theory. Finally, as all proposals are based on a complete truth, there is general dissent about Kuipers's idea that this instantiation-explication distinction requires the incompleteness of the truth. Both our modal and monadic paraphrase of Kuipers's approach, clearly show that the modal intuitions underlying Kuipers's theory representations do not require an incomplete truth; the distinction between instantiation and explication has nothing to do with the completeness of the truth. The empirical truth of a language is complete iff all contingent propositions have fixed empirical truth-values—that is, iff the language is *determinate*. The incompleteness of the truth is not a necessary condition for the modal implementation of physical theories.

4.6. SUMMARY AND PROSPECTS

4.6.1. Summary

In the current chapter, I presented and analysed three answers to the epistemic problem of approach to the truth. Additionally, I sketched some technical revisions that need further investigation.

1. According to the consequence definition, which reads "only consider the true consequences", failure to falsify a strong and improbable empirical theory corroborates a relatively large step towards the truth.
2. The irreversibility of *Kuipers's* content and likeness rules, "choose the most successful theory", is conditional on the truth of the strongest accepted law $S(t)$. Generally, Kuipers's likeness rule is not functional for approaching the truth; however, for convex theories, with the truth included, it is.
3. *Niiniluoto* proposes a probabilistic, reversible rule. He chooses as an estimate for the truthlikeness of a theory, $\mathscr{E}(\mathrm{Tr}(\mathbf{H},g))$, the mathematical expectation of the random variable $\mathrm{Tr}(\mathbf{H},g)$.
4. The three proposals fulfill *different adequacy conditions*: *Popper's* wants the rules to determine "under what conditions what hypotheses should be selected *for tests*" (his italics);[57] *Niiniluoto*'s rule rationally reconstructs scientific progress; and *Kuipers's* rule explains the lasting empirical success of scientific theories.
5. The epistemic proposals of Niiniluoto and Kuipers both need *technical revision.*
   a. The *structuralist rules* need a stratified (modal or monadic) representation to parry the paradoxical consequences regarding the logical strength of R and S; the condition of the empirical truth of law S is better replaced by a probabilistic rule.
   b. Regarding quantitative problem sets, the rule of *expected truthlikeness*, $\mathscr{E}[\mathrm{Tr}(g,\mathbf{H})]$, may disagree with the rules based on $\Delta(\mathscr{E}[\mathrm{H}], g)$, the distance between an answer and the expectation of the complete answers (prop. 4.11 p. 158).
6. The structuralist proposals display more *congruence between rule and definitions* than the method of expected truthlikeness.
   a. The content definition and rule are both based on logical consequences and the likeness definition and rule on similarity between structures.
   b. Niiniluoto's truthlikeness *definition* is based on similarity between structures, whereas his *rule* adjusts its preference order on true consequences e conditionalizing the underlying probability distribution.
7. Niiniluoto's ver-rule and Kuipers's success rule have both their *problems of application.* Given a body of evidence only very view theories are compared by the content rule; and the ver-rule presupposes complete knowledge of all logical possible answers to a problem and an appropriate probability distribution for all those complete answers.
8. All rules assume that the language underlying the comparison is determinate, and the truth is complete. Consequently, both rules fail to choose between theories if the language is indeterminate.

4.6.2. Prospects

In the present chapter, I gave a detailed account of Kuipers's and Niiniluoto's answers to the epistemic problem of approach to the truth, and their specific properties. The next step in the research, however, concerns more general investigations into the relation between these specific answers and similar endeavours in adjacent fields, especially those branches of logic involved in the 'dynamic turn'.[58] For example, there is a strong correspondence between non-monotonic reasoning such as belief revision, conditional logic and preferential reasoning, and the answer to the epistemic problem.[59] Such a general comparison, therefore, requires a systematic investigation into the *logical properties* of the answers to the epistemic problem considered in the present chapter.

We have already encountered the question whether the rule had to be *(ir)reversible*, that is whether, in the light of new evidence, it may reverse a (strict) preference order. A related logical property is the soundness of a preference relation. A system of preferences is *sound* iff *the preferred theory is at least as good as the original one*; moreover, it is *progressive* iff the preferred theory is better. In other words, the rule always points in the right direction. A second related logical property is the *completeness* of the rule. A rule of preferences is *complete* iff picking out theories from the set of all logically possible ones, in the light of the incoming evidence, eventually, the rule will bring us to the truth. In other words, the rule will not get stuck in "dead ends" or have halting problems, e.g. circles, or infinite long paths. The two properties mentioned need not be logically independent. For (finite) propositional languages, a progressive rule is complete, since the set of all logically possible theories is finite. Obviously, the implication does not hold for first order languages.

Additionally, we must consider the computational properties of the rules. A rule of theory-choice may be called *decidable* iff it decides between any pair of theories, no matter the evidence. Further, we may consider the *complexity* of the rules; one rule of theory-choice may be computationally much more intricate than another. Of course we would prefer efficient rules, which provide outcomes within a reasonable time span. Obviously, these two properties are related. The decidability of a rule counterbalances its complexity. We already encountered this contrast. Kuipers's content rule is easily decidable but not complete; and Niiniluoto's rule of expected truthlikeness is complete but far too complex for practical purposes. Kuipers's and Niiniluoto's proposals, however, are two extremes on a continuum of possible solutions. The interesting question is now whether there is some optimum for decidability and complexity between the approaches of Kuipers and Niiniluoto.

CHAPTER 5

THE HIDDEN VARIABLE

I will discuss the alleged "language-dependency" of truthlikeness definitions in this chapter. When I introduced the differences between content and likeness definitions in the first chapter, I briefly mentioned the possible change of a truthlikeness order after extensional substitutions. Now it is time to present and examine Miller's objection to truthlikeness definitions.[1] This chapter has the following outline. After the introduction, in Section 5.2 I introduce Miller's puzzle and describe the reactions it provoked. In Section 5.3 I will discuss the formal analysis of the translation puzzle; and in the fourth section I formulate the solution. I end the chapter with a summary and a list of results.

5.1. INTRODUCTION

We saw that Miller and Tichý both published their discovery of the flaw in Popper's verisimilitude definition in 1974.[2] Miller also criticized Tichý's alternative definition for being "language-dependent" in the same paper. According to Miller, a truthlikeness order that is not preserved under extensional substitutions, is "language-dependent," and therefore "demonstrably false."[3] Miller's criticism triggered a "language-dependency" debate. For example, according to Niiniluoto, additional requirements are needed to pass on a truthlikeness order from one language to another.[4] He doubts whether there are semantical conditions that can guarantee the "same truthlikeness order under translation." Pragmatic considerations are likely to enter the argument. Despite the various answers to Miller's criticism, the problem has not been solved yet: witness Barnes (1991).

In this chapter I introduce Miller's argument, and consider it under the strongest interpretation possible. My answer is radically different from those proposed so far. Miller's observation about truthlikeness definitions is correct, and although he presents it as a curse, it proves to be a blessing. If one distinguishes between the language of the cognitive problem, and that of the formulation of the theories, Miller's argument shows that truthlikeness is a relative, rather than an absolute notion.

5.2. MILLER'S OBJECTION TO THE "COUNTING METHOD"

I introduce Miller's extensional substitution argument against truthlikeness definitions as far as propositional languages are concerned in this section. First I shall introduce Miller's objection, and sketch some of its consequences. Then, I describe two intuitive foundations of the truthlikeness notion that come under fire in Miller's attack. I then deal with the solutions found in the literature, and finally, I show that the refined structuralist proposals also fall prey to Miller's argument.

5.2.1. Miller's Argument of Extensional Substitution

Our introduction of Miller's extensional substitution argument will be as short as possible and therefore it is presented in a slightly different manner than the original.[5] To start with, let 'Anglo-Saxons' use a "rudimentary weather language $\mathscr{L}$", to describe the weather conditions. $\mathscr{L}$ consists of:

1. the logical machinery of propositional logic and
2. the non-logical vocabulary 'h','r','w',

which designate the basic propositions 'it is hot', 'it is rainy' and 'it is windy', respectively. All Anglo-Saxons agree that the next propositions adequately represent the weather conditions on the first, second, and third day of September.

(1) $X = \neg h \wedge \neg r \wedge \neg w$, *the first day*
$Y = \neg h \wedge r \wedge w$, *the second day*
$T = h \wedge r \wedge w$, *the third day.*

Recall that we neglect the syntactic differences of equivalent $\mathscr{L}$-sentences and represent propositions in the disjunctive normal form (dnf); for instance we do not distinguish between $\neg(h \vee r) \wedge \neg w$ and X.

Further, we say that the weather has changed smoothly or *continuously* if the conditions of yesterday agree with the conditions of today on more literals than the conditions of the day before yesterday do. Else, if the conditions of yesterday agree with the conditions of today on *less* literals, it is said that the weather changed *discontinuously*. In our example, the weather conditions of Y differ from those of T only in the temperature dimension whereas the conditions of X and T differ in all relevant dimensions. Consequently, all Anglo-Saxons agree that over the first three days of September the weather changed *continuously*. The same intuition underlies the naive definition of truthlikeness for propositional languages. Consider X, Y and T to be theories, and let T be the true theory. Then, it is plausible to say that Y is closer to the truth T than X for the very same reason. X differs from T in more basic dimensions than Y differs from T.

Miller, however, rejects the preceding "counting method" because it lacks robustness.[6] To illustrate his point, he proposes translating X, Y and T into X', Y' and T' using another rudimentary weather language $\mathscr{L}'$, which also has three atomic propositions: h, m and a. The three transformation formulae (MP) of (2) fix the relation between $\mathscr{L}$ and $\mathscr{L}'$:

$$\text{(2)} \qquad \begin{aligned} h &:=_{tr} h, \\ m &:=_{tr} h \leftrightarrow r, \\ a &:=_{tr} h \leftrightarrow w \end{aligned}$$

Miller calls $\mathscr{L}$ and $\mathscr{L}'$ *intertranslatable*, and proposes to call 'm', 'it is Minnesotian' and 'a', 'it is Arizonian'.[7] Formulae (2) are used to transform the propositions of (1) into

$$\text{(3)} \qquad \begin{aligned} X' &= \neg h \wedge m \wedge a, \\ Y' &= \neg h \wedge \neg m \wedge \neg a, \\ T' &= h \wedge m \wedge a \end{aligned}$$

Clearly, this extensional substitution does not preserve the naive truthlikeness order. Counting the differences between Y' and T' and those between X' and T' leads to the conclusion that X' is closer to the true theory T' than Y' although $X \equiv X'$, $Y \equiv Y'$ and $T \equiv T'$.[8] People using language $\mathscr{L}$ conclude that the weather of the first three days of September changed gradually whereas other people, using $\mathscr{L}'$, conclude the opposite; the changes in the weather during the same period and at the same place, were *dis*continuous.[9]

Miller's h,r,w-argument only takes one and a half page in print. Consequently, he did not discuss all the relevant questions in depth. At least two of them are indispensable for a good understanding of the present example. The first one is: does (2) comprise object or metaformulae, and the second concerns the semantics of $\mathscr{L}$ and $\mathscr{L}'$.

I will answer the object or metaformulae problem first. Miller's objection does not hold if (2) are object formulae. Since then, X', Y' and T' are sentences of an extended *h-r-w-m-a* language $\mathscr{L}_{\text{ext}}$ in which (2) are meaning postulates; and as the naive definition only takes into account basic literals, Miller's transformation does not affect the original truthlikeness order. Miller, therefore, takes (2) to be *metaformulae*: witness Tichý's (1978) note four. The metaformulae do not increase the non-logical vocabulary of $\mathscr{L}$, neither do they introduce m and a in $\mathscr{L}$. The formulae only guarantee identical truth-values of related sentences in corresponding possible worlds. According to Miller $\mathscr{L}$ and $\mathscr{L}'$ have the same power of expression since the transformation defines a bijection between their constituents.

The second question to be asked is whether the meanings of $\mathscr{L}$ coincide with those of $\mathscr{L}'$. Having observed that the transformation formulae (MP') for $\mathscr{L}'$ are

$$\text{(4)} \qquad h :=_{tr} h$$

$$r :=_{tr} h \leftrightarrow m$$
$$w :=_{tr} h \leftrightarrow a$$

Miller claims "perfect symmetry" between $\mathscr{L}$ and $\mathscr{L}'$.[10] Comparison of (2) and (4) supports the symmetry thesis between $\mathscr{L}$ and $\mathscr{L}'$ on the syntactic level. On the semantic level, however, this symmetry is compatible with two mutually exclusive interpretations of the languages.

According to the first interpretation, corresponding *possible worlds remain separate entities*, and (2) does not connect the (extensional) meanings of $\mathscr{L}$ and $\mathscr{L}'$. Note that as $\mathscr{L}$ and $\mathscr{L}'$ are different languages, in principal they have different semantics, and the logically possible worlds depend on the vocabulary of those languages. Since the substitution, or "translation", of $\neg h \wedge r \wedge w$ yields $\neg h \wedge \neg m \wedge \neg a$, the $\mathscr{L}$-possible world $\langle 0,1,1 \rangle$ relates to the $\mathscr{L}'$-possible world $\langle 0,0,0 \rangle$, they do not coincide.

Secondly, we may interpret (2) such that the extensions of the $\mathscr{L}$-sentences are *identical* to the extensions of the corresponding $\mathscr{L}'$-sentences. According to this interpretation, the $\mathscr{L}$-possible world $\langle 0,1,1 \rangle$ is identical to the $\mathscr{L}'$-possible world $\langle 0,0,0 \rangle$ since the "translation" of $\neg h \wedge r \wedge w$ is $\neg h \wedge \neg m \wedge \neg a$. Identification of $\mathscr{L}$- and $\mathscr{L}'$-extensions means that a translation from $\mathscr{L}$ to $\mathscr{L}'$ does not change the original meanings. Later publications imply that Miller favours the second interpretation; he acknowledges: "there is a problem of explaining under what conditions this [that is: sentences in different languages having the same assertive power (*SZ*)] obtains."[11]

Finally, it should be noted that the influence of Miller's example goes beyond the scope of truthlikeness. It is also relevant for comparison and assessment of empirical success of a theory. Hence, empiricists should also take notice of Miller's example. Moreover, the argument applies to forms of logic in which the likeness between possible worlds plays an important role. Among others, many forms of non-monotonic reasoning; conditional logic (counterfactuals) and preferential reasoning are important examples.

To summarize my introduction of Miller's objection, I tentatively conclude that, as far as extensional interpretations are concerned, (2) comprises *metaformulae identifying* the corresponding possible worlds of $\mathscr{L}$ and $\mathscr{L}'$.

5.2.2. Two Intuitions

Miller's argument affects two ideas underlying the approach-to-the-truth project. I will discuss the notion of an *independent truth* first and then I will say something about the idea of an *objective approach* to the truth.

In the pre-Tarskian era, the borderline between *truth* and *reality* was vague. C.S. Peirce, one of the first to write about the modern idea of approach to the truth, stressed the difference between the two. The following quotation shows, however, that, according to Peirce, besides its property-like character, truth is autonomous and language independent. Moreover, he seemed to hold, following Kant, that Truth is a *regulative* idea. After many years of experimenting, science will ultimately succeed in approaching the Truth.

"Truth is a character which attaches to an abstract proposition, such as a person might utter. It essentially depends upon that proposition's not professing to be exactly true. But we hope that in the progress of science its error will indefinitely diminish, just as the error of 3.14159, the value given for π, will indefinitely diminish as the calculation is carried to more and more places of decimals." (C.S. Peirce (1965, 5.565))

This intuitive idea of Truth I shall call the *citadel conception* of Truth. It has the following characteristics. Truth has an independent character, and is only obtainable by hard labour. There are many roads leading to the same Truth possibly formulated in different languages. It is a rather vague metaphysical notion primarily ascribed to propositions. No wonder adherents of this conception of truth needed to emphasize the difference between reality and truth.

The next quotation demonstrates Popper's insight that Tarski's truth definition may replace the preceding metaphysical notion of truth.

"Yet whenever I used to write, or say, something about science getting nearer to the truth, or as a kind of approach to the truth, I felt that I really ought to be writing 'Truth', with a capital 'T', in order to make quite clear that a vague and highly metaphysical notion was involved here, in contradiction to Tarski's 'truth' which we can with a clear conscience write in the ordinary way with small letters. It was only quite recently that I set myself to consider whether the idea of truth involved here was really so dangerously vague and metaphysical after all. Almost at once I found that it was not, and that there was no particular difficulty in applying Tarski's fundamental idea to it." (Popper (1972, p 231-232))

In contrast with the citadel conception of Truth, I shall call Tarski's systematic definition of truth, *truth-in-$\mathscr{L}$*. Its characteristics are: truth-in-$\mathscr{L}$ can only be defined on the basis of one specific formal language $\mathscr{L}$; if voc($\mathscr{L}$) consists of observable notions, such as found in Miller's example, the truth-value of a sentence is relatively easy to obtain; reformulation of the same truth must use the same formal language $\mathscr{L}$; truth is an exact notion and it is a property of sentences.

Popper proposed to replace the *independent* truth notion in Peirce's approach-to-the-truth with Tarski's *dependent* truth-in-$\mathscr{L}$, and as adequate translations preserve truth-values and deduction relations, translation of the truth does not affect a verisimilitude, content, order.[12] As under the same translation truthlikeness, likeness, orders may change, Miller's argument shows that truthlikeness proposals break with the tradition of an independent, absolute, truth.

The second idea that increases the impact of Miller's objection is that approach to the truth must be an *objective* notion. Adequate translations must preserve truth-values, and Tarski's truth definition provides an *objective* notion of truth. Similarly, Popper wanted verisimilitude to be an objective notion, and Miller's argument seems to undermine the objectivity of truthlikeness proposals. According to Miller, truthlikeness devaluates into a helplessly *subjective* notion, as it allows for a reversal of the order after "translations." Evidently, the citadel notion of Truth goes hand in hand with the objectivity constraint. The objectivity restriction also causes the quantitative truthlikeness definitions to attract a lot of attention. In addition to an increase of strength, quantification appears to add objectivity. Even if two theories are couched in different languages, we may compare their truthlikeness as both are comparable to the, objective, truth.

5.2.3. Four Kinds of Responses

Miller put forward his ingenious objection more than two decades ago, and since then many scholars have reacted to his criticism. None of the responses claims that his argument is *formally* incorrect, but there is less unanimity about its consequences. I divide the answers to Miller's argument into four categories, note that the classification is neither unique nor complete.

First, one may consider Miller's argument to be a proof of the *impossibility of an objective truthlikeness definition*, resulting in the collapse of the whole truthlikeness enterprise. For example, Urbach states:

"I shall argue, moreover, that *the attempt to make sense of an objective notion of degrees of closeness to the truth for false theories is fundamentally and irretrievably misguided.*" (Urbach (1983) p. 267, his italics)

Urbach argues that "there is no unique, absolute, objective sense in which two different structures are more or less like some third structure."[13] Barnes also states: "'truthlikeness' cannot supply a basis for an objective account of scientific progress. .... objectivity of scientific progress must be grounded on the fact ... that knowledge, not mere truth, is the aim of science."[14] Barnes concludes "... the entire project at which they (Tichý, Oddie, Niiniluoto, Miller, etc.) have laboured is, to some extent, just misconceived."

The second kind of reaction takes truthlikeness to be a more serious subject of investigation. It claims that the *h-r-w* language epistemologically precedes the *h-m-a* one. Verifying X precedes verifying X' because under regular circumstances observing 'it is dry' precedes observing 'if it is dry, then it is hot; and if it is hot, then it is dry.' Consequently, the *h-r-w* order is the correct one. It seems that Miller's statement:

"...there seems no good reason — beyond sheer prejudice — for treating the *h-r-w* language as more fundamental than the *h-m-a* one,"

points to this objection of epistemological asymmetry.[15] Generally, this *privileged language argument* states that language $\mathcal{L}$ is more useful to the investigator than language $\mathscr{L}'$, which is derived from $\mathscr{L}$.[16] The argument reads that scientists do not readily, without good reason, give up their research vocabulary as it is an important tool for investigation. Miller's argument would be more serious if the interpretation of scientific vocabulary consisted exclusively of conventions.[17]

For example, Mormann accuses Miller of holding "*negative essentialism*" (his italics) since Miller does not realise that "many properties are part of physical magnitude and that the meaning of such a term is not exhausted by simple formal definition."[18] Brink and Heidema bring forth similar arguments, they claim "... there is indeed good reason —beyond sheer prejudice— to choose between the *h-r-w* language and the *h-m-w* language." For a given world, scientists "choose one language over another as being *more appropriate* to that world (their italics)."[19] In private discussions, Kuipers also uses similar arguments to parry Miller's example.

The privileged language argument deprives truthlikeness of its objectivity. A native tribe using the *h-m-a* vocabulary can perfectly do without the conceptions of hot, rainy and windy. They can discriminate the same kinds of weather states as $\mathscr{L}$-speaking people. Moreover, if *Martians* had *sense data h, m, a*, and used them to describe weather states, the human state descriptions would be epistemologically *less* adequate for them than their own. To reject Miller's argument because of its epistemological asymmetry is to render truthlikeness dependent of the human apparatus of perception. Consequently, the language-dependency argument cannot save the objectivity of truthlikeness. At best it remains an intersubjective notion.[20]

The third kind of response to Miller's problem *scrutinizes the term 'translation'* for Miller's interpretation of (2). A translation is not some syntactic transformation rule that provides a bijection between the sentences of $\mathscr{L}$ and $\mathscr{L}'$, and between the valuations of the languages. A good translation must preserve meanings.

Tichý and Oddie claim that (2) does not provide a translation, and deny the identity of the possible worlds of $\mathscr{L}$ and $\mathscr{L}'$, they base their ideas on those of Rudolf Carnap, who differentiates between the *interpretation or meaning* and *models*.[21] According to Tichý and Oddie, worlds are "assignments of extensions to certain non-syntactic items, namely traits or attributes."[22] Accordingly, "a logical space is a collection of worlds." Since the logical space of the $\mathscr{L}$-valuations differs from the space of the $\mathscr{L}'$-valuations, (2) is not a translation. X or Y paraphrase propositions and a proposition dichotomizes a logical space. A sentence and its translation ought to refer to the same items, and since $\mathscr{L}$ and $\mathscr{L}'$ establish different logical spaces, T is not a translation of T'.[23] This argument has the drawback of leaving the truthlikeness research in the realms of intensional interpretations.

The fourth possible response takes a more moderate point of view in between the extremes of Miller and Oddie. According to Miller, truthlikeness has to be immune to any extensional substitution. Tichý and Oddie, however, want truthlikeness to be immune only for intensional translations. The more moderate position in between those two extremes asks for reasonable additional requirements that preserve a truthlikeness order under extensional "translations". For example, using a first-order formulation of the problem, David Pearce concludes that making an appeal to intensional languages "fails to resolve the problem, or contribute to its clarification."[24] Pearce asks the more moderate question: what requirements can we add to the semantics of a translation to preserve a truthlikeness order? Pearce formulates his question in algebraic terms: what subset of homeomorphisms between the models of $\mathscr{L}$ and $\mathscr{L}'$ preserves a truthlikeness order? Niiniluoto paraphrases the additional requirement; and is pessimistic whether this question can be solved in semantic terms.

"...truthlikeness should be preserved in a translation from $\mathscr{L}$ to $\mathscr{L}'$ *if the cognitive problem does not change* within this language shift. ... the choice of a proper language is at least partly a function of our cognitive interests which in turn depend on pragmatic conditions."[25]

5.2.4. The Refined Structuralist Definition[26]

Before presenting my analysis of Miller's argument, I want to investigate whether the objection also affects the refined semantic proposals of the structuralists. Most truthlikeness definitions in the literature paraphrase scientific theories with sets of propositions, and Miller's argument applies to them all.[27] The structuralist approach, however, might be an exception, since it represents a theory as a set of possible worlds. As the naive structuralist proposals ignore similarity between possible words, they evade Miller's argument; therefore the question reads: do the refined structuralist definitions also escape Miller's criticism? I shall answer this question for the refined qualitative and quantitative versions.

First, we check the refined *qualitative* approach. Let the propositions h, r and w, designate: 'It is hot', 'It is raining' and 'It is windy.' Next, let the theories X, Y, and T be the singletons containing: $x = \neg h \wedge \neg r \wedge \neg w$, $y = \neg h \wedge r \wedge w$, and $t = h \wedge r \wedge w$, which are elements of the set of all conceptual possibilities, M_p. Let $s(x,y,t)$ mean: y is at least as similar, or close, to t as x. Thus, x, y and t are constituents, and $s(x,y,t)$ boils down to: on all literals about which x agrees with t, y also agrees with t; and $r(x,t)$ means that x and t are *related* or *comparable*. Further, recall we saw that the structuralist refined qualitative truthlikeness reads as follows: RTL(X, Y, T) iff

$$\text{(Ri)}\quad \forall x \in X, \forall t \in T \mid r(x,t) \rightarrow \exists y \in Y : s(x,y,t)$$
$$\text{(Rii)}\quad \forall y \in Y-(X \cup T) \mid \exists x \in (X-T), \exists t \in T : s(x,y,t)$$

Clearly, X, Y and T fulfil the clauses (Ri) and (Rii). There is just one $x \in X$ and there is just one $t \in T$. They are comparable, they consist of the same number of basic propositions, and y intermediates x and t since for all literals, if x agrees with t, than so does y. This complies with (Ri). Concerning the second clause, (Rii), y is the only element in $Y-(X \cup T)$, x the only one in $(X-T)$ and t the only one in T. These three elements meet the condition s(x,y,t). Conclusion:

(5) RTL(X, Y, T)

Let (2) translate theories X, Y and T, into the singletons X', Y' and T' containing the elements $x' = \neg h \wedge m \wedge a$, $y' = \neg h \wedge \neg m \wedge \neg a$ and $t' = h \wedge m \wedge a$, respectively. Just as in the previous case x', y' and t' are comparable; but now the tables are turned for x' and y'. Structure x' agrees on more literals with t' than y'. The remainder of the checkout is analogous to the previous one, and therefore:

(6) RTL*(Y', X', T').

The asterisk points to the $\mathscr{L}'$-dependency of the truthlikeness order. The conclusion reads: Miller's argument reverses the structuralist refined qualitative definition.

Secondly, we must check whether the *quantitative* refined definition also falls prey to Miller's argument. The definition is formulated in terms of distances between sets of structures. First, let us consider the preliminaries: let d(x,y), the distance between the structures x and y, be the number of literals on which x and y disagree:

(7) $\mathrm{d}(x,y) :=_{def} |\{\text{mismatch of literals}\}|$, and let
(8) $\mathrm{d}(x,Y) :=_{def} \min\{\mathrm{d}(x,y) \mid y \in Y\}$

define the distance between an individual structure and a set of structures, d(x,Y). The distance from X to Y, d($X \backslash Y$), fixes the distance between X and Y, RTD(X, Y).

(9) $\mathrm{RTD}(X,Y) :=_{def} \mathrm{d}(X \backslash Y) + \mathrm{d}(Y \backslash X)$ in which $\mathrm{d}(X \backslash Y) := \Sigma_x \, \mathrm{d}(x,Y)$,

The RTD-distance between X, Y equals the "counting method". Evidently, according to (9), in the present situation, Y is closer to the truth T then X; $\mathrm{d}(X,T) = \mathrm{d}(T \backslash X) + \mathrm{d}(X \backslash T) = 2\mathrm{d}(\{\neg h \wedge \neg r \wedge \neg w\}, \{h \wedge r \wedge w\}) = 6 > 2 = 2\mathrm{d}(\{\neg h \wedge r \wedge w\}, \{h \wedge r \wedge w\}) = \mathrm{d}(T \backslash Y) + \mathrm{d}(Y \backslash T) = \mathrm{d}(Y,T)$; therefore:

(10) $\mathrm{RTD}(X,T) \geq \mathrm{RTD}(Y,T)$

Again, the second step in our argument is the (2)-translation of X, Y and T into (3). After this transformation

(11) $\mathrm{RTD}^*(Y',T') \geq \mathrm{RTD}^*(X',T')$ obtains since

$\mathrm{d}(Y',T') = \mathrm{d}(T' \backslash Y') + \mathrm{d}(Y' \backslash T') = 2\mathrm{d}(\{\neg h \wedge \neg m \wedge \neg a\}, \{h \wedge m \wedge a\}) = 6 > 2 = 2\mathrm{d}(\{\neg h \wedge m \wedge a\}, \{h \wedge m \wedge a\}) = \mathrm{d}(T' \backslash X') + \mathrm{d}(X' \backslash T') = \mathrm{d}(X',T')$. Again, the asterisk suggests the $\mathscr{L}'$-dependency.

Let me summarize the results of the present subsection. Despite their semantic theory representation, the refined structuralist proposals cannot keep out of the range of Miller's artillery. The refined structuralist orderings are not preserved under existensional substitutions. To apply Miller's argument, I took the relevant structures to be truth-value assignments that verify constituents. In these circumstances the qualitative and quantitative structuralist refinements are based on the "counting method", and extensional substitutions may affect their orderings. This corroborates my claim that the refined structuralist proposals are truthlikeness definitions, whereas the naive content definitions are verisimilitude proposals.

5.3. MODELLING MILLER'S "TRANSLATION"

Now that Miller's argument and several reactions have been presented, I will look at some formal features of the argument. First, I introduce some terminology, including the adequacy condition for translations borrowed from Pearce and Rantala. Then, I shall consider the valuation table of the various constituents of $\mathscr{L}$ and $\mathscr{L}'$; and finally I will consider the relation between the two constituent algebras of $\mathscr{L}$ and $\mathscr{L}'$.

5.3.1. Some Terminology

Generally, a translation systematically correlates syntactic entities of a source language to those of a target language, in accord with their respective semantics. In this section I elaborate this contention for propositional languages.

Let σ and ρ denote the syntactic and the semantic part of the translation, respectively. The *function* σ maps the set of syntactic entities of $\mathscr{L}$ into that of $\mathscr{L}'$, and ρ is the with-σ-corresponding *function* that maps the set of semantic entities of $\mathscr{L}'$ onto that of $\mathscr{L}$. Since σ is into, $\mathscr{L}'$ has at least as much expressive power as $\mathscr{L}$. The reason for defining ρ from $\mathscr{L}'$ onto $\mathscr{L}$ is that if $\mathscr{L}'$ has at least as much expressive power as $\mathscr{L}$, then every $\mathscr{L}$-sentence will relate to one translation in $\mathscr{L}'$; but on the level of the valuations, every valuation of $\mathscr{L}$ may relate to more than one valuation of $\mathscr{L}'$. Consequently, ρ is conveniently defined as a function from the semantic level of $\mathscr{L}'$ onto that of $\mathscr{L}$.

The following notational conventions denote some formal features of Miller's objection.[28]

$\mathscr{L}$:= the source language, i.e. the translated language
$\mathscr{L}'$:= the target language, i.e. the result of the translation
σ (σ') := the syntactic translation function from $\mathscr{L}$ ($\mathscr{L}'$) to $\mathscr{L}'$ ($\mathscr{L}$)
V (V') := an $\mathscr{L}$ ($\mathscr{L}'$)-valuation or possible world of $\mathscr{L}$ ($\mathscr{L}'$)
ρ (ρ') := a function on the valuations of $\mathscr{L}'$ ($\mathscr{L}$) to valuations of $\mathscr{L}$ ($\mathscr{L}'$)

Miller defines his "translation" on a syntactic level, and specifies σ by (4), and σ' by (2); he does not discuss the *adequacy* of his "translation", but simply assumes that it is adequate regarding the semantics of both languages. Miller assumes that the valuation verifying one $\mathscr{L}$-constituent c corresponds to the $\mathscr{L}'$-valuation that verifies $\sigma(c)$. Formally, the adequacy requirement may be formulated thus:

ADE: σ is *adequate* with regard to ρ iff for all φ of $\mathscr{L}$ and all V' of $\mathscr{L}'$:
$$V' \vDash \sigma(\varphi) \Leftrightarrow \rho(V') \vDash \varphi,$$

ADE explicates the following intuition about accurate translations.[29] For all sentences φ of the source language, if its translation is true at some possible world V', then φ is also true in the corresponding possible worlds $\rho(V')$; and the same obtains for the converse. In other words, for all $\mathscr{L}$-valuations V' and all $\rho(V')$ of $\mathscr{L}'$, the truth-values of all source sentences and their translations must coincide. Thus, ρ guarantees correspondence of meaning.

Our notational conventions enable us to express Miller's claim about perfect symmetry. Let us consider the pair of translations σ and σ'. As stated before, σ is assumed to be adequate regarding ρ. The first claim of symmetry is that σ' is also adequate regarding ρ'. In other words, ADE' obtains too:

ADE′: σ' is *adequate* w. r. t. ρ' iff for all φ' of $\mathscr{L}'$ and all V of $\mathscr{L}$
$$V \vDash \sigma'(\varphi') \Leftrightarrow \rho'(V) \vDash \varphi'$$

The similarity between ADE and ADE′ is obvious. The second symmetry claim is that σ and σ' are mutual inverses, just as ρ and ρ'. Consequently, the following translation prescription also obtains for Miller's example:

$$\varphi \equiv \sigma'(\sigma(\varphi)) \quad \text{and} \quad \varphi \equiv \sigma(\sigma'(\varphi))$$
$$V = \rho(\rho'(V)) \quad \text{and} \quad V' = \rho'(\rho(V'))$$

Note that, generally, in more interesting situations of translation or reduction, ρ is a many-one function.[30] It is only in the special situation of Miller's translation that σ and ρ are bijections.

5.3.2. A Table of Valuations

In the preceding subsection we encountered some terminology used to denote specific elements of the example of extensional substitution. In this subsection I put the constituents and valuation of $\mathscr{L}$ and $\mathscr{L}'$ into one truth-table. Table 1 shows the main syntactic and semantic characteristics of Miller's objection. Column (ii) shows the possible valuations of the atomic propositions of $\mathscr{L}$.

The order of the table is based on a systematic truth-value assignment to h, r and w. Every row relates to one valuation, and column (i) displays the $\mathscr{L}$-constituent that corresponds to that valuation. For example, G is true if it is cold, dry and windy, and the truth-value assignment of h, r and w on the fourth row is $\langle 0,0,1 \rangle$, therefore G resides on row four. If we interpret σ and σ' to be metaformulae of $\mathscr{L}_{\text{meta}}$, we can deduce in $\mathscr{L}_{\text{meta}}$ that G' is the translation of G.[31] On each row, the constituent in column (i)′ is the σ-counterpart of the constituent in column (i); but how must (ii)′, the fourth column of the table, be filled in, if the assignments verify the translations of the constituents of column (i)?

Table 1. Miller's 1974-objection

	(i)	(ii)	(ii)′	(i)′
	True constituent in state of affairs described in $\mathcal{L}$	$V \equiv \rho(V')$ $\langle h,r,w\rangle$	V' $\langle h,m,a\rangle$	True constituent in state of affairs described in $\mathcal{L}'$
1	$T = h \wedge r \wedge w$	$\langle 1,1,1\rangle$	$\langle 1,1,1\rangle'$	$T' = h \wedge m \wedge a$
2	$Y = \neg h \wedge r \wedge w$	$\langle 0,1,1\rangle$	$\langle 0,0,0\rangle'$	$Y' = \neg h \wedge \neg m \wedge \neg a$
3	$C = h \wedge \neg r \wedge w$	$\langle 1,0,1\rangle$	$\langle 1,0,1\rangle'$	$C' = h \wedge \neg m \wedge a$
4	$G = \neg h \wedge \neg r \wedge w$	$\langle 0,0,1\rangle$	$\langle 0,1,0\rangle'$	$G' = \neg h \wedge m \wedge \neg a$
5	$B = h \wedge r \wedge \neg w$	$\langle 1,1,0\rangle$	$\langle 1,1,0\rangle'$	$B' = h \wedge m \wedge \neg a$
6	$F = \neg h \wedge r \wedge \neg w$	$\langle 0,1,0\rangle$	$\langle 0,0,1\rangle'$	$F' = \neg h \wedge \neg m \wedge a$
7	$D = h \wedge \neg r \wedge \neg w$	$\langle 1,0,0\rangle$	$\langle 1,0,0\rangle'$	$D' = h \wedge \neg m \wedge \neg a$
8	$X = \neg h \wedge \neg r \wedge \neg w$	$\langle 0,0,0\rangle$	$\langle 0,1,1\rangle'$	$X' = \neg h \wedge m \wedge a$

To begin with, the first elements of the (ii)′-valuations are identical to the first elements of the valuations of column (ii). The second and third places of the (ii)′-valuations also have to be filled in accordance with (4). Formally, if the translation is adequate, the (ii)′-valuations have to be ρ-related to the elements of column (i). Consequently, to fill in (ii)′ correctly, we must first take one truth assignment of *h*, *r* and *w*, e.g. [4, (ii)]: $\langle 0,0,1\rangle$; then, we must take the constituent that is true for that assignment (*G*), and relate it by σ to G', correctness of the propositional calculus, and thus of $\mathcal{L}_{\text{meta}}$ warrants $\sigma(G) \equiv G'$; and finally, presupposing that Miller's "translation" is adequate, we must give the truth assignment of *h*, *m* and *a* for which $\sigma(G)$ is true (at [4, (ii)′]: $V' \equiv \langle 0,1,0\rangle$).

Table 1 also illustrates another distinction that we heave mentioned already. In section 5.2, I characterized two ways of supplementing the semantics of $\mathcal{L}$ and $\mathcal{L}'$. In the first place, following Tichý and Oddie, we may claim that the valuations of $\mathcal{L}$ *differ* from the valuation of $\mathcal{L}'$. I agree with Tichý and Oddie that, logically speaking, this is the most natural analysis. In this situation, the translation is adequate regarding ρ iff $\langle 0,0,0\rangle = \rho\langle 0,1,1\rangle'$, and ρ *relates* different valuations, allowing for a change of meaning. Secondly, I claimed that the strongest possible interpretation of Miller's argument must assume that the valuations of $\mathcal{L}$ and $\mathcal{L}'$ are *identical*.[32] For example the second row claims that $\langle 0,0,0\rangle = \langle 0,1,1\rangle'$. We have seen that an extended language containing both $\mathcal{L}$ and $\mathcal{L}'$ can establish this identity.

Finally, I sketch the semantics of $\mathcal{L}$ and $\mathcal{L}'$ for the strongest interpretation of Miller's example. Let $\mathcal{L}$ and $\mathcal{L}'$ be two sublanguages of the extended language $\mathcal{L}_{ext}(h,r,w,m,a)$, and let $\mathcal{L}_{ext}$ plus (2) (or (4)) define $\mathcal{L}$ (or $\mathcal{L}'$). Both languages have the same subset of possible valuations of $\mathcal{L}_{ext}$, although $\mathcal{L}$ restricts those models to $\{h, r, w\}$, and $\mathcal{L}'$ restricts them to $\{h, m, a\}$. Thus, $\mathcal{L}$ and $\mathcal{L}'$ yield the same means of expression as they produce the same sets of possible worlds. In other words:

$$\forall V \in \mathrm{Mod}(\mathcal{L}_{ext}) \colon V \vDash MP \text{ (p. 170)} \;\Leftrightarrow\; V \vDash MP' \text{(p. 170)}$$

Thus, in Miller's example, two scientific communities may come to use the two different sublanguages of $\mathcal{L}_{ext}$; and since these communities do not know $\mathcal{L}_{ext}$ or (2), they will order the mentioned theories differently.

5.3.3. The Constituent Algebras

It is interesting to consider the impact of Miller's transformation on the syntactic structure of the other constituents. After all, the syntactic form of a constituent establishes its distance to the truth. Miller's bijection between the constituent algebra of $\mathcal{L}$ and that of $\mathcal{L}'$ is shown in Figure 1. In the next chapter, we shall extensively encounter the notion of a constituent algebra. For the moment it suffices to know that a constituent algebra is isomorphic to a set algebra of the vocabulary of $\mathcal{L}$. The constituent corresponding to some subset A of atomic sentences is the conjunction of the sentences in A and the negations of the sentences in A^c. Figure 1 shows the constituent algebras of $\mathcal{L}$ and $\mathcal{L}'$, and illustrates that Miller's substitution bijection between the algebras is not order preserving.

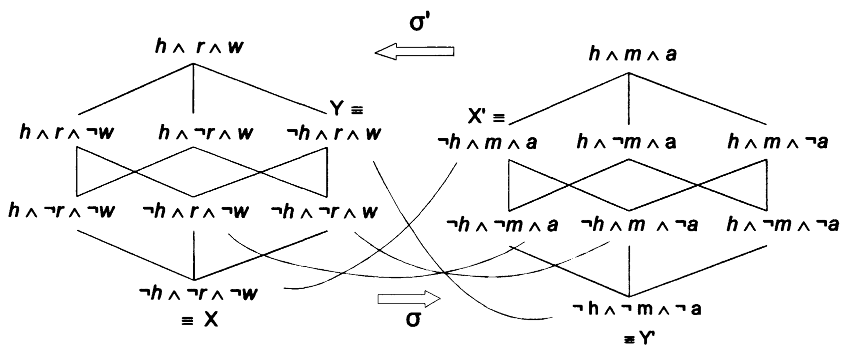


Fig. 1. The bijection between the constituent algebras

Figure 1 depicts the syntactic form of all the constituents of $\mathcal{L}$ and $\mathcal{L}'$. Theory X has the same syntactic form as Y'; the same holds for Y and X'. All the basic propositions of X and Y' are preceded by a negation sign whereas in Y and X' only the first proposition is preceded by this sign. The syntactic form of F and G also changes after being subjected to σ', and the form of G' and F' corresponds to those of F and G, respectively. Although the structure of F and G changes under the transformation, their distances to the truth remain the same. F and G do not pose a problem for the "counting method" in the way X and Y do. Table 1 and Figure 1 show that Miller's transformation only changes the truthlikeness order of sets that contain X or Y.

5.4. THE SOLUTION TO MILLER'S PROBLEM

According to Miller, every truthlikeness definition based on the "counting method" is demonstrably false. It violates the intuition that "different formulations of the same theory ought to have the same distance to the truth;" but are Miller's conclusions, even under the strongest interpretation, unavoidable? Definitely not. In this section, I give the solution to the problem by pointing out that there is *hidden variable* in Miller's argument. As so often happens in philosophical discussions, it is not the formal paraphrase that contains the error, but it is the interpretation of the formalities that causes the trouble. I shall present my solution in three stages. First, in the next subsection, I propose interpreting (2) semantically. The example given in the second subsection shows that a change of order is legitimate after an extensional substitution of the traits underpinning that order. Then, by *distinguishing two functions of the vocabulary*, in the third subsection, I present the solution to Miller's puzzle, which was interestingly foreshadowed by Hilpinen (1976). Truthlikeness definitions are protected against Miller's substitution objection if we make explicit the list of the traits underlying the similarity relation. We shall see that this list of traits is the hidden variable in the so-called "language dependency" debate. Finally, in the fourth subsection I shall discuss the contribution my solution makes to Niiniluoto's notion of a cognitive problem.

5.4.1. The Semantical Interpretation of MP

In the preceding sections, we saw that Miller interprets (2) syntactically: σ establishes a "translation" between $\mathcal{L}$ and $\mathcal{L}'$. I argue that it is only *this syntactic interpretation* that gives Miller's example its paradoxical ring. A *semantical* interpretation of (2) provides a better perspective from which to assess the effects of the extensional substitution. As the formal apparatus of the example is propositional logic, $\mathcal{L}$ and $\mathcal{L}'$ are correct, complete and decidable languages. Consequently, the syntax and semantics are different sides of the same coin, and it is at least as feasible to assess (2) by its semantic effects.

Consider the strongest possible interpretation of Miller's example (Section 5.2). Assume that σ is adequate regarding ρ *if and only* if ρ *identifies* valuations of $\mathscr{L}$ with valuations of $\mathscr{L}'$; then it is clear why Miller's result is not counter intuitive or even strange. *Several objects*, in our case possible worlds, *receive a different order, if classified by different sets of properties*; and the mutual translatability of those sets of properties does not affect this truism. This is the first important ingredient of our solution to the extensional substitution problem. More specifically, in Miller's case, the objects to be ordered are the eight valuations of $\mathscr{L}$ and $\mathscr{L}'$, and the different sets of properties according to which the valuations are ordered are the mutually translatable vocabularies of $\mathscr{L}$ and $\mathscr{L}'$. The next example shows that the semantic interpretation invalidates Miller's objection, and that the change of order after extensional substitution has to be welcomed.

5.4.2. An Example

The following is an example in which human beings receive a different similarity order after the traits underlying the likeness relation are replaced by other properties. It shows that Miller's change of order is *desirable* within an extensional shift of language.

Let $\mathscr{L}^{\text{phen}}$ and $\mathscr{L}^{\text{gen}}$ be monadic predicate languages formulating phenotypical and genotypical facts about genetics.[33] The domain that gives $\mathscr{L}^{\text{phen}}$ and $\mathscr{L}^{\text{gen}}$ its interpretation is the set of all human individuals with colour-blind offspring. Let Table 2 define the vocabularies of $\mathscr{L}^{\text{phen}}$ and $\mathscr{L}^{\text{gen}}$; and let $M(x)$ and $Cb(x)$ abbreviate 'x is male' and 'x is colour-blind', respectively. Next, let us consider the following elementary facts about sex-linked traits.

Table 2. The vocabularies of $\mathscr{L}$ and $\mathscr{L}'$

$\mathscr{L}^{\text{phen}}$: phenotypic	$\mathscr{L}^{\text{gen}}$: genotypic
$M(x)$:= 'x is male'	$XY(x)$:= 'x has exactly one X and one Y chromosome in its body cells'
$Cb(x)$:= 'x is colour-blind'	$X^c(x)$:= 'x has exactly one recessive colour-blindness allele in its body cells'

The numbers of X and Y chromosomes in the body cells determine the sex of a human being. A man has one X and one Y chromosome and a woman has two X and no Y chromosomes in her body cells. Roughly, alleles on the chromosome are responsible for inheritable traits. Chromosomes occur in pairs, and a recessive trait becomes manifest only if the responsible alleles occur on both chromosomes, otherwise the dominant trait will become manifest. Some recessive traits such as colour-blindness and haemophilia are linked to genes of the X chromosome. The Y

chromosome contains only few genes, which primarily relate to the maleness of the person. Consequently, in a male, most traits of the X chromosome, including recessive traits, will develop as the related Y chromosome does not contain the dominant counterpart.

Why do I restrict the domain of our language to people with colour-blind offspring? The answer is that females in this group must have at least one colour-blindness allele in their body cells. The X^c chromosome of a son must come from the female as the Y chromosome comes from the father. If a female has a colour-blind daughter, then she has also given the daughter a colour-blindness allele. Consequently, for people with colour-blind offspring, $\neg X^c(x)$ means that the number of X^c chromosomes, containing a colour-blindness allele, is zero for a man and is two for a woman. Finally, $\neg XY(x)$ equals $XX(x)$, $XX(x)$:= 'x has exactly two X chromosomes,' since not being a man is identical to being a woman.

The preceding facts about genetics result in the following (extensional) "translation" from $\mathscr{L}^{\text{phen}}$ to $\mathscr{L}^{\text{gen}}$

$$M(x) :=_{tr} XY(x)$$

In words: a person 'is a man' if and only if '(s)he has one X and one Y chromosome'. The second rule reads

$$(12) \qquad Cb(x) :=_{tr} X^c(x) \leftrightarrow XY(x)$$

In words: a person, with colour-blind offspring, 'is colour-blind' if and only if '(s)he has in their body cells a Y chromosome and an X chromosome that contains the colour-blindness allele' or '(s)he has two X chromosomes and both have the colour-blind allele'. In other words: a person, with colour-blind offspring, 'is colour-blind' if and only if 'all her/his X chromosomes contain the colour-blindness allele' Equation (12) obtains, since, as we saw, females from our domain with normal sight have exactly one X chromosome with the colour-blindness allele in their body cells.

Let us turn to Miller's proposal of extensional substitution. Let a, b and z be three individuals with colour-blind offspring; then the following three identities obtain:

(13)	$\neg M(a) \wedge \neg Cb(a)$	$\Leftrightarrow$	(16) $\neg XY(a) \wedge X^c(a)$
(14)	$\neg M(b) \wedge Cb(b)$	$\Leftrightarrow$	(17) $\neg XY(b) \wedge \neg X^c(b)$
(15)	$M(z) \wedge Cb(z)$	$\Leftrightarrow$	(18) $XY(z) \wedge X^c(z)$

Thus, the answer to the question whether a or b is more like z depends on the decision whether the question is meant phenotypically or genotypically. Phenotypically, b, the colour-blind woman of (14) is more similar to the colour-blind man z of (15) than a, the woman of (13) who can see all colours. Genotypically, person b of (17) with two X chromosomes, both containing the recessive colour-blindness allele, is *less similar* to person z of (18) with one X chromosome containing the colour-blindness allele, than the person a of (16), with two X chromosomes of which only one contains the colour-blindness allele. In other words, as far as 'XY' and '*having exactly one* colour-blindness' are concerned, X^cX^c is less similar to X^cY than X^cX, the third agrees with the second on the number of X^c. Using Niiniluoto's terminology, we may say that the cognitive problem in $\mathscr{L}^{\text{phen}}$ differs from the one in genotypical $\mathscr{L}^{\text{gen}}$.

$\mathscr{L}^{\text{phen}}$: "Is a or is b more like z regarding colour-blindness and sex?"
$\mathscr{L}^{\text{gen}}$: "Is a or is b more like z regarding the number of X^c's and the combinations of X and Y?"[34]

Our genetic example illustrates that similarity of objects is a relative affair, and possible substitutions are irrelevant. The 'likeness' of one element to another depends on a set of traits underpinning the similarity relation. Consequently, if objects are classified by the vocabulary of $\mathscr{L}^{\text{phen}}$, a change of order in a shift to $\mathscr{L}^{\text{gen}}$ is not surprising, even if the terms of the first are definable in terms of the second. On the contrary, it is plausible and even desirable that similarity relations disagree if based on different sets of traits. Our example shows that the demand for conservation of a truthlikeness order under changing cognitive interests is unreasonable.

Miller is likely to disagrees with the preceding statement. He takes the side of David Lewis:

"Do we not endlessly make statements of comparative similarity, among motor cars, people, faces and all sorts of other things that are intended to be more or less independent of the choice of primitive vocabulary? If I say 'Hans takes after his mother more than his father', I should not withdraw my judgement when it is pointed out that, with regard to some properties, he is more like his father." (*private correspondence*).

In my opinion, all our comparisons of objects are implicitly or explicitly based on a list of relevant traits. This becomes apparent when we disagree about a similarity order. Then, opponents come up with traits that verify their similarity order. Additionally, the disagreement will not be exclusively about *which* properties are involved it will be also about how much it contributes to the total comparison. In this respect, I agree with N. Goodman, who claims that "comparative judgements of similarity often require ... selection of relevant properties" and in the more sophisticated cases "a weighing of their relative importance."[35] Although in a more complex way, even D. Lewis claims that

"Overall similarity consists of innumerable similarities and differences in innumerable *respects of comparison*, balanced against each other according to the relative importance we attach to *those respects of comparison* (my italics)."[36]

Finally, we also find Hilpinen on our side. In the third chapter, we saw that the likeness of the possible worlds is related to an *independently* chosen characteristic C^i. Following David Lewis, the first paper on truthlikeness defines a system of nested spheres $\mathcal{N}^i_u$ around a world $u \in U$, *which depends on the characteristic* C^i.

Let us summarize the results obtained so far. The most important aspect of the strongest interpretation of Miller's argument is the assumption that for all $c \in$ Const($\mathcal{L}$) and $\sigma(c)$ the ρ-related valuations are *identical* (subsections 5.2.1 and 5.3.2). Usually, the traits in the vocabularies of $\mathcal{L}$ and $\mathcal{L}'$ are taken to underpin the similarity between the valuations. In the present section, I showed that as the similarity order depends on *the choice of the underlying traits*, extensional substitution of the traits may change the similarity order.

5.4.3. The Solution: Two Functions of the Vocabulary

We have seen that a semantic analysis of Miller's example gives a better perspective on the language dependency discussion than Miller's own syntactic point of view. One consequence of the semantic viewpoint is the insight that the substitution argument is an instance of a very down-to-earth phenomenon: the similarity order of objects, in Miller's case: valuations, depends on the choice of the properties according to which the objects are ordered.

The discussants in the language dependency debate have overlooked a second feature of Miller's example. This second feature is the subject of the present subsection and provides the most important ingredient for our solution to the extensional substitution problem. We need to realize that in truthlikeness definitions the non-logical vocabulary *has two totally different functions*. The first function is the *identification* of the various possible worlds or constituents. In $\mathcal{L}[h,r,w]$, $T := h \wedge r \wedge w$ differs from $Y := \neg h \wedge r \wedge w$, but with respect to $\mathcal{L}[h, r]$, T and Y are equivalent. The second function of the vocabulary consists of *providing the set of traits* according to which the constituents are ordered. For that reason, if one claims that $Y = \neg h \wedge r \wedge w$ is more like $T = h \wedge r \wedge w$ than $X = \neg h \wedge \neg r \wedge \neg w$, one tacitly assumes that the h,r,w-triple is the means of comparison. Miller's objection shows that X is more like T than Y if instead of the h,r,w-triple, one takes the h,m,a-triple as a means of comparison. Neglecting the two different functions of the non-logical vocabulary, we cannot avoid Miller's paradoxical conclusions. The hidden variable of *the language dependency debate, therefore, is the set of traits that underpins the order of the possible worlds.* It is important to note that this analysis *avoids* the metaphysical *essentialism*, which is typical of the privileged language argument. It is our cognitive interest that establishes the traits of our concern, but we do not claim anything about the *status* of those traits. More particular, they need not be more fundamental aspects of reality than other properties.

The upshot of the distinction between the two functions of the basic linguistic elements is twofold. First, it shows that the conclusion: "truthlikeness based on the 'counting method is language dependent", is a misleading description of how Miller's extensional substitution affects a truthlikeness order. It is far more accurate to say that the similarity between theories depends on a particular set of traits, often fixed by the logical vocabulary. Second, it is a fruitful distinction since it helps us to delineate the abstract idea of a cognitive problem. This is the subject of the next subsection. Here, we will elaborate on the independent choice of the ordering traits in Miller's example.

Usually, truthlikeness definitions use the vocabulary of the theories to order the theories. The genetics example shows that this choice is not obligatory. An adequate description of the extensional substitution requires a list of traits that determines the similarity between the theories. The following notation makes this list explicit. Thus (h, r, w): $\neg h \wedge r \wedge w \ \leq_{h \wedge r \wedge w} \ \neg h \wedge \neg r \wedge \neg w$ denotes: "Regarding h, r, w, $\neg h \wedge r \wedge w$ is closer to the truth $h \wedge r \wedge w$ than $\neg h \wedge \neg r \wedge \neg w$." (h, r, w) specifies the list of relevant traits, the context, or even the relevant cognitive problem, as we shall see shortly. Using the new notation, we distinguish between

(19)	(h, r, w): $\neg h \wedge r \wedge w \ \leq^L_{h \wedge r \wedge w} \ \neg h \wedge \neg r \wedge \neg w$
(19′)	(h, r, w): $\neg h \wedge \neg m \wedge \neg a \ \leq^L_{h \wedge r \wedge w} \ \neg h \wedge m \wedge a$
(20)	(h, m, a): $\neg h \wedge \neg r \wedge \neg w \ \leq^L_{h \wedge r \wedge w} \ \neg h \wedge r \wedge w$
(20′)	(h, m, a): $\neg h \wedge m \wedge a \ \leq^L_{h \wedge r \wedge w} \ \neg h \wedge \neg m \wedge \neg a$

in which $\leq^L_{h \wedge r \wedge w}$ is a likeness order based on the "counting method" and the theories are

$$\begin{aligned} X &:= \neg h \wedge \neg r \wedge \neg w = \neg h \wedge m \wedge a \\ Y &:= \neg h \wedge r \wedge w = \neg h \wedge \neg m \wedge \neg a \\ T &:= h \wedge r \wedge w = h \wedge m \wedge a. \end{aligned}$$

For instance, (19) claims that regarding the h,r,w-triple Y is more like T than X. Formula (2) transforms (19) into (19′), and Miller maintains that the second is incompatible with (20′), the likeness order of $\mathscr{L}'$. Our analysis shows, however, that the four situations are all different. Although the means of expression of (19) and (19′) diverge, they concern the same cognitive problem. The same obtains for (20) and (20′). The difference between (19) and (19′) on the one hand, and (20) and (20′) on the other, is that they concern different cognitive problems.

Let us explain the situation in terms of Section 5.2: If X, Y, and T are the first second and third day of September, then the corresponding claims are:

(19)	The (h,r,w)-weather conditions, expressed in h,r,w-terms, of the first three days of September changed *continually*
(19′)	The (h,r,w)-weather conditions, expressed in h,m,a-terms, of the first three days of September also changed *continually*

(20) The (h,m,a)-state conditions, expressed in h,m,a-terms, of the first three days of September changed *discontinuously*

(20′) The (h,m,a)-state conditions, expressed in h,r,w-terms, of the first three days of September also changed *discontinuously*

Our analysis solves the problem of the alleged incompatibility of (19′) and (20′). According to Miller, the four statements must display the same order. Niiniluoto disagrees, and claims that extensional substitutions may change a cognitive problem. Explicating the hidden variable, the set of traits that orders the theories, our analysis distinguishes between the cognitive problem of (19) and of (20). There is nothing "demonstrably wrong" or even paradoxical about the conjunction of $\langle h, r, w\rangle$: $Y \leq^L_h X$ and $\langle h, m, a\rangle$: $X \leq^L_h Y$; the two relations concern different cognitive interests. Additionally, our analysis agrees with Hilpinen (1976) who claims that the choice of the relevant traits indexes the set of nested spheres.

5.4.4. Different Cognitive Problems

Niiniluoto claims that a translation should not alter a truthlikeness order if it does not change the cognitive problem.[37] This claim added to the supposition that Niiniluoto's truthlikeness definition is sound, implies that Miller's substitution *changes the cognitive problem*. His formal representation of the cognitive problem, however, does not reflect this change.

As mentioned in Chapter 3, Niiniluoto defines the concept of a cognitive problem using a problem set **B**.[38] Let **B** be a set of mutually exclusive and jointly exhaustive elements, relative to background knowledge b. Then, the cognitive problem regarding issue **B** reads:

(21) Which element of **B** is true?

In Miller's example, **B** and **B**′ are the sets of all $\mathscr{L}$ and $\mathscr{L}'$-constituents. Transformations σ and σ' establish bijections between **B** and **B**′. The question arises, if σ is a truth-preserving bijection between **B** and **B**′ how Niiniluoto can refer to a difference between (21) and (21′).

(21′) Which element of **B**′ is true?

Translating **B** does not alter the problem since the substitution preserves all truth-values. According to Niiniluoto's definition, there is no difference between cognitive problems **B** and **B'** if the second is the result of extensionally substituting the terms of the first. The question now reads: what are the criteria for two cognitive problems to be identical?

The answer to the problem relates to our distinction between the linguistic means of comparison, and the vocabulary used to formulate the theories. This distinction gives the opportunity to refer to the traits of our cognitive interest. *It is the traits of our cognitive interest that identify the cognitive problem.* Our proposal reads that a cognitive problem P is identical to problem P' if, after a substitution σ, the truth-values are preserved for all relevant valuations, *and the vocabulary of our cognitive interest remains the same.*[39] In subsection 5.2.3, we saw that Miller's example prompted the question: what requirements do we have to add to the semantics of a translation to preserve a truthlikeness order? This problem has a definite, but disappointingly trivial answer. No extensional substitution changes a truthlikeness order *if it remains based on the same vocabulary of our original cognitive interest.*

The preceding discussion showed that Niiniluoto's formulation "truthlikeness should be preserved in a translation from $\mathcal{L}$ to $\mathcal{L}'$ *if the cognitive problem does not change*" is pointing in the right direction. His general definition of a cognitive problem, however, is too general to solve Miller's challenge.[40] Our analysis of Miller's argument considers two elements. There is nothing wrong with the translation of the vocabularies of the theories, as in (19) and (19′). Although the vocabulary of the theories changes, the set of traits of our cognitive interest, and therefore the cognitive problem remains the same. Consequently, it does not affect the more natural order of the theories. An additional transition, however, of the relevant traits, as from (19′) to (20′), *changes the cognitive problem*, and restores the original order in $\mathcal{L}'$. Our genetics example showed that this change of order is legitimate and even indispensable.

5.5. SUMMARY AND PROSPECTS

Let us summarize the results of the present chapter and sketch some of prospects for future research.

5.5.1. Summary

1. At the end of his 1974-paper, while criticizing Tichý's proposal in the same issue of the British Journal, Miller argues that an approach-to-the-truth proposal has to be invariant under extensional substitutions.
2. Miller's substitution argument, however, fails to prove that all truthlikeness definitions that come down to the "counting method" are too strong.[41] I refuted Miller's argument in three steps.

a. From the semantical point of view, Miller must *identify* the possible worlds related to $\mathcal{L}$ and to $\mathcal{L}'$, or else his argument collapses at the start.

b. Similarity among those possible worlds depends on a *list of traits* according to which the worlds are ordered, and there is no *absolute* similarity.

c. Without distinguishing the *descriptive* from the *ordering function* of a vocabulary, an *extensional substitution may change the preference ordering* as the substitution changes the cognitive problem, see my genetic example.

3. The possible difference between the order of the $\mathcal{L}$-theories, and $\mathcal{L}'$-theories that are translated into $\mathcal{L}$ is unproblematic, since the "translated" theories concern a *different cognitive problem*. Thus, the original order (h, r, w): $\neg h \wedge r \wedge w \leq^{L}_{h \wedge r \wedge w} \neg h \wedge \neg r \wedge \neg w$, reads, after h,m,a-substitution: (h, r, w): $\neg h \wedge \neg m \wedge \neg a \leq^{L}_{h \wedge r \wedge w} \neg h \wedge m \wedge a$. This order is not at odds with (h, m, a): $\neg h \wedge m \wedge a \leq^{L}_{h \wedge r \wedge w} \neg h \wedge \neg m \wedge \neg a$ since this clearly concerns a *different* cognitive problem.
4. It would be useful for Niiniluoto to make explicit the basic properties of a cognitive problem. In its present form, a cognitive problem only involves picking out the true element of a P-set, and the definition disregards the underlying vocabulary.
5. The answer to Pearce's question, what kind of extensional language shift will preserve a truthlikeness order, becomes rather trivial in the setting of this chapter. An extensional shift of vocabulary preserves the truthlikeness order if the set of the traits of our cognitive interest remains the same.

5.5.2. Prospects

To parry Miller's substitution argument in the propositional case, the cognitive problem must specify the traits underpinning the order. It remains to be seen whether this method also throws new light on the quantitative and the predicate versions of Miller's objection.[42] The prospects are good since they are variations on the same theme.

Miller's argument is not restricted to truthlikeness definitions. With some minor adjustments it applies to many logical systems based on a preference relation on possible worlds, such as systems for conditional logic (counterfactuals), preferential reasoning, belief revision, and for many other forms of non-monotonic reasoning. Future research will show how hard Miller's argument hits in those areas, and what impact our hidden variable analysis may have.

From the point of view of natural languages, Miller's argument raises the question of which logical or even pragmatic relations between source and target sentences need to be preserved by good translations. It seems plausible that all deduction relations in the source language need to be preserved in the target language. The first question that arises, then, is whether this relation is symmetric; in other words: do all deduction relations in the target language also need to be present in the source language; and what do we have to think about more subtle implications such as the Gricean conversational implicature, and all sorts of presuppositions?[43] Are they also supposed to be invariant under adequate translations?

Furthermore, is keeping deduction relations in place always more important than keeping the pragmatic implications? Or does this depend on the goal of the text? Translations of proverbs and poems suggest that transferring the "idea" might counterbalance the more formal constraints of what is supposed to be a good translation. The point is that it remains to be seen whether a good translation must in all cases guarantee an isomorphism between the topologies of the logical deduction relation in source and target language. Yet, this is the supposition on which Miller has build his argument.

The refutation of Miller's extensional substitution argument in the current chapter paves the way for our proposals of the final chapter. There, I shall show that the existing content orderings are compatible with plausible likeness orderings, and propose a way to merge them. This approach yields a new comparative approach-to-the-truth definition, *refined verisimilitude*, which leads naturally to a plausible quantitative proposal. The definition of refined verisimilitude combines most advantages of the content and likeness approaches, and avoids a considerable part of their drawbacks.

CHAPTER 6

REFINED VERISIMILITUDE

The difference between verisimilitude and truthlikeness definitions is presented in Chapter 1; I then expounded on this difference in Chapters 2–3. Together, the first three chapters constitute the expository part of this book. In Chapters 4–5, I discussed the epistemic problem of approach-to-the-truth, and Miller's extensional substitution argument. Putting forward the solution to Miller's puzzle, in the preceding chapter, I paved the way for my refined verisimilitude proposal, which is the subject of this chapter. In a way, Chapters 1–5 can be viewed as preparatory steps leading up to my new approach-to-the-truth proposal presented in Section 6.4. As a prelude I first introduce the refined Δ-verisimilitude definition that merges the Δ-distances on the Lindenbaum and constituent algebra of a language. Then we will see that the $\leq^+$ ordering of the $\mathcal{L}$-propositions can be combined with the total preorder on the constituents of the language. I shall partition the Lindenbaum algebra in equivalence classes of propositions of the same distance to the truth, and I shall prove that the merger of the two orderings is compatible with this partition and orders the equivalence classes. I call the resulting ordering the *refined verisimilitude* proposal. A more detailed summary of this chapter can be found at the end of Section 6.1.

6.1. INTRODUCTION

Let us consider the general complaint of the truthlikeness proponents regarding the Δ-definition. Niiniluoto (1987, p 192) formulates this shortcoming thus:

"The fundamental weakness of all definitions of truthlikeness—among them Popper's original proposal and Miller's reformulations of it—that are based on the symmetric difference Cn(h) Δ **T** is simply the fact that they don't pay any attention to the underlying metric structure of the space of complete answers. It is not the *size* of the falsity content that matters, but rather the distance of its elements from the truth. Popper's and Miller's qualitative theories of verisimilitude ... fail to handle false theories, because they don't include tools for measuring the degrees of 'nearness to the truth'". (Niiniluoto 1987 p. 192).

Of course, Miller knows that his content definition neglects the topology of the complete answers. He uses the terms "horizontal" and "vertical improvement" to refer to this problem.

"[V]ertical improvement ... boils down ... to an increase in logical and empirical content". "[H]orizontal improvement ... is a question of fit. ... the extent to which one hypothesis can improve in accuracy on another." (Miller (1994) p. 220).

Regarding $\mathscr{L}[p,q]$, $p \leq_{p\wedge q} p \vee q$ is an example of vertical, and $p \wedge \neg q \leq_{p\wedge q} \neg p \wedge \neg q$ of horizontal improvement. Miller's content proposal concerns vertical improvement and since he is entangled in various versions of his "language dependency" argument, he remains sceptical about all definitions of horizontal improvements.

The *goal* of merging the content and likeness ordering into refined verisimilitude is threefold. In the first place, it proves that the *absence of likeness considerations in content definition is an accidental*, and not a structural affair; our refined verisimilitude is a genuine content proposal that considers likeness between constituents. The second goal is to introduce a new approach-to-the-truth definition that incorporates likeness between possible worlds *and avoids the arbitrary way in which current truthlikeness proposals lift the preferences between constituents to weaker propositions.*[1] The last goal is to show that refined verisimilitude has many desirable properties.

To begin with, an important question reads: how does my refined verisimilitude proposal copes with the *child's-play argument*? Concerning the non-modal version, the answer reads: although refined verisimilitude softens the child's-play objection, the conjunction of two serious but false propositions is still closer to the truth than one of its conjuncts. Not all equally strong antecedences of a false theory are, however, uncomparable. Our likeness refinement orders the false antecedences of a false theory. Yet, the complete solution to the child's-play problem requires a modal strategy. Fortunately, since refined verisimilitude is a content proposal, it is open to the modal strategy presented in Chapter 2. Consequently, the modal treatment of our refined verisimilitude proposal looks promising, and only shortage of space and time forces us to leave the details to another occasion.

Besides the child's play and the behaviour under extensional substitution, I mentioned *truth-value dependency* as an important property of verisimilitude and truthlikeness proposals (Chapter 1). Recall that an approach-to-the-truth order $\leq_\tau$ is truth-value dependent iff

$$\forall\varphi,\psi \in \mathrm{Prop}(\mathscr{L})\text{: if } \tau \vDash \psi \text{ and } \varphi \vDash \neg\tau\text{, then } \varphi \not\leq_\tau \psi.$$

We saw that all published qualitative approach-to-the-truth definitions of the preceding chapters were truth-value dependent. According to all of them no false theory is better than the tautology. Miller (1974) acknowledges that the Δ-definition, just as the consequence definition does, suffers from the child's-play argument. He does not mention, however, the big advantage of the $\leq^+$-ordering, namely its truth-value *in*dependence. The Cn-proposal is the only qualitative approach-to-the-truth proposal according to which a false theory can be better than a true theory. That is the reason for me to replace the Δ-proposal with the +-definition in the second half of this chapter. If we stick to the content proposal of Miller and Kuipers, then the refinement would remain truth-value dependent.

This chapter consists of three parts. In the first part, Section 6.2, I shall introduce the basics of my likeness refinement technique. A one-one function maps the *constituent algebra* $\mathscr{A}$ onto the atoms of the Lindenbaum algebra $\mathscr{B}$; and the order of the constituent algebra induces a likeness order on all elements of $\mathscr{B}$, the terminology used will be explained in due course. This horizontal likeness order, $\preceq_h$, turns out to be compatible with the vertical $\leq^{\Delta}$-order, and their combination, the refined Δ-verisimilitude definition, results in the $\leq^{\Delta}_{\precsim}$-order of Prop($\mathscr{L}$). The subscript h indicates the *logical strength* of the ordered propositions.

Since the $\leq^{\Delta}_{\precsim}$-order is still truth-value dependent, in the second part, Sections 6.3–6.4, I replace the $\leq^{\Delta}$-order by the $\leq^{+}$-order, and use algebraic means to examine the $\leq^{+}$-order. It turns out to be homomorphic to the Lindenbaum algebra $\mathscr{B}$. In Section 6.4, I strengthen the $\preceq_h$ likeness order into $\precsim_h$, which orders the atoms of $\mathscr{B}$; $\precsim_h$ is a total preorder of $\mathscr{B}$. Next the preorders $\leq^{+}$ and $\precsim_h$ both induce partitions on $\mathscr{B}$: $\mathscr{B}/\sim$ and $\mathscr{B}/\simeq$. The important result of this chapter reads that *the disjunctive closure of* $\leq^{+}$ *and* $\precsim_h$, $\leq^{+}_{\precsim}$ *partially orders* $\mathscr{B}/\approx$, which is the finest partition of $\mathscr{B}$ yielded by $\mathscr{B}/\sim$ and $\mathscr{B}/\simeq$. This refined verisimilitude ordering $\leq^{+}_{\precsim}$ is the first truth-value independent qualitative content order.

In the third and final part of this chapter, Section 6.5, I introduce the quantitative version of the refined verisimilitude ordering, $\sqsubseteq$. It is truth-value independent, but not weakly context independent. The main achievement of the current chapter is the proof that the combined order $\leq^{\Delta}_{\precsim}$ of Section 6.2 implies the (stronger) order $\leq^{+}_{\precsim}$ of Section 6.4; whereas the $\sqsubseteq$-order of Section 6.5 is again a quantitative strengthening of the one of Section 6.4. Textbox 1 contains a visualization of the outline of the present chapter. Note that the "⇒" signs represent the material implication;[2] and "⊕" symbolizes merging of ordering relations.

Constituent Algebra		Lindenbaum Algebra				
$\mathscr{A}$	⊕	$\mathscr{B}$	⇒	$\leq^{\Delta}_{\precsim}$ (VI.2)		
⇓		⇓ (*homomorphism*)		⇓		
$\mathscr{A}/\equiv$ (6.4)	⊕	$\mathscr{B}/I_{\tau}$ (6.3)	⇒	$\leq^{+}_{\precsim}$ (VI.4)	is $\leq$ of $\mathscr{B}/\approx$	
				⇓		
				$\sqsubseteq^{+}$ (VI.5)		

Textbox 1. The outline of Chapter 6.

I shall not examine combinations represented by the "diagonals" of the diagram. It would unnecessary lengthen the chapter and only yield predictable results. I have aimed at immediate generalisations merging the two weak orders $\preceq_h$ and $\leq^\Delta$ into $\leq^\Delta_{\preceq}$, and the two strong orders $\precsim_h$ and $\leq^+$ into $\leq^+_{\precsim}$. The two qualitative mergers and the quantitative refinement of the last section, $\sqsubseteq^+$, form a chain of ordering relations of increasing strength.

We have also encountered *specularity* as a metatheoretical property of verisimilitude and truthlikeness proposals (Chapter 1.). In the current chapter the specularity property plays a subsidiary role. In the first place, it is easy to see that the +-definition is not specular. For instance, $p \not\models q \vee (p \wedge q)$ and $\neg p \models \neg q \vee (\neg p \vee \neg q)$ (see p. 56); and as all truth-value independent proposals of the present chapter are based on the +-definition, the truth-value independent definitions of the present chapter are not specular.

The *language dynamic behaviour* was the third property we dealt with in Chapter 1. We considered the invariance of an order under extensions of the language. More specifically, I mentioned context independency and, in Chapter 3 logical unbiasedness (p. 98). In the sections that are to come, I shall repeat the relevant definitions, and observe that these properties are interestingly distributed over the various approach-to-the-truth proposals of this chapter.

6.2. REFINED Δ-VERISIMILITUDE: $\leq^\Delta_{\preceq}$

This section is dedicated to a provisional version of my refined verisimilitude proposal introduced in section 6.4. First, I will introduce the required terminology; then I present a qualitative likeness definition; and finally I will introduce and examine a merger of the Δ-definition and my proposal to compare the truthlikeness of propositions using the similarity among their models.

6.2.1. Constituent and Lindenbaum Algebras

In this subsection, I consider the constituent and the Lindenbaum algebra and their relation. First, let me introduce a Boolean algebra as a special kind of lattice, which is considered to be a special kind of order. Consider the following reminder.

Reminder: ρ is a *preorder* iff ρ is a reflexive, transitive relation; ρ is a *partial order* iff ρ is an antisymmetric preorder.

A preorder ρ on A induces a partial order on the set of equivalence classes $[a] \subset A$ defined by $\forall b \in A$: if $a \, \rho \, b$ and $b \, \rho \, a$ then $b \in [a]$. Further, a is uncomparable with b regarding the strict partial order $<$ on A, $a \not\asymp b$, iff $a \neq b$, $a \nless b$ and $b \ngtr a$. Let $\leq$ be a partial order in P. Element $p \in P$ is the *upper bound* of $A \subseteq P$ if $x \leq p$ for all x in A; if, in addition, p is the *smallest* upper bound of A, $p \leq p'$ for all (A-) upper bounds $p' \in P$, then p is the *least* upper bound (lub) of A. The *greatest lower bound* (glb) of A is the dual of the glb of A and is defined similarly.

Reminder: $\mathscr{L}$ is a *lattice* iff $\mathscr{L}$ is partially ordered and all pairs $\langle a, b\rangle$ have a lub, $a \vee b$, and a glb, $a \wedge b$. A lattice is *distributive* iff $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ and $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$. $\mathscr{B}$ is a *Boolean Algebra* iff $\mathscr{B} \supset \{0,1\}$ is a distributive lattice, and each $a \in \mathscr{B}$ has an unique complement a' such that $a \wedge a' = 0$, $a \vee a' = 1$. Consequently, $a \leq b$ iff $a \vee b = b$ iff $a \wedge b = a$ iff $a \wedge b' = 0$.

The foregoing definitions imply that a Boolean algebra $\mathscr{B}$ is a special kind of order. Every propositional language $\mathscr{L}$ is related to a Boolean algebra. I define the *Lindenbaum* algebra of propositions algebraically.[3]

DEFINITION 6.1: $\mathscr{B}_{\mathscr{L}} := \langle P, \wedge, \vee, ', 1, 0\rangle$ is *a Lindenbaum Algebra* of $\mathscr{L}$ iff
1. $\mathscr{B}_{\mathscr{L}}$ is a Boolean algebra
2. $P = \{ [\varphi] \mid \varphi \in \text{Sent}(\mathscr{L})\}$ and $[\varphi] :=_{def} \{\psi \in \text{Sent}(\mathscr{L}) \mid \psi \equiv \varphi\}$
3. $[\varphi] \wedge [\psi] := [\varphi \wedge \psi]$, $[\varphi] \vee [\psi] := [\varphi \vee \psi]$, and $[\varphi]' := [\neg\varphi]$
4. $1 = [\top]$, and $0 = [\bot]$

The definition implies $[\varphi] \leq [\psi]$ iff $\varphi \models \psi$. In most of the following I assume that $\mathscr{L}$ is a finite (propositional) language. $\mathscr{L}$ is finite if its set of non-logical vocabulary, voc($\mathscr{L}$), is finite. I designate $\mathscr{L}$-propositions by φ, ψ, χ_1 ..., χ_n.[4] In the finite propositional case, constituents are the atoms of $\mathscr{B}_{\mathscr{L}}$; they are the strongest non-contradictory propositions. Constituents C_i have the following form:

$$C_i := \bigwedge_{p \in C} p \wedge \bigwedge_{q \notin C} \neg q$$

for some $C \subset \text{voc}(\mathscr{L})$. Therefore, arbitrary sets of atomic propositions correspond to constituents in $\mathscr{B}_{\mathscr{L}}$. Consequently, it makes sense to consider a second Boolean algebra $\mathscr{A}_c$.

DEFINITION 6.2: The Boolean set algebra $\mathscr{A}_c := \langle \wp(\text{voc}(\mathscr{L})), \cap, \cup, {}^c, \varnothing, \text{voc}(\mathscr{L})\rangle$ is the *constituent algebra* of $\mathscr{L}$.

The elements of $\mathscr{A}_c$, which are sets of atomic propositions, are designated by C_1, ... C_n, $n = 2^{|\text{voc}(\mathscr{L})|}$. Thus a finite propositional language yields two distinct Boolean algebras; the Lindenbaum algebra $\mathscr{B}_{\mathscr{L}}$ and the constituent algebra $\mathscr{A}_c$. The next mapping designates the relation between the two. The one-one function f: $\mathscr{A}_c \mapsto \mathscr{B}_{\mathscr{L}}$ defined by

$$f(C) := \bigwedge_{p \in C} p \wedge \bigwedge_{q \notin C} \neg q \qquad (\text{e.g. in } \mathscr{L}[p,q]\text{: } f(\{p\}) = p \wedge \neg q)$$

maps the elements of the constituent algebra $\mathscr{A}_c$ onto the atoms of the Lindenbaum algebra $\mathscr{B}_{\mathscr{L}}$ which I refer to as α, β, γ. I refer to the set of constituents implying a proposition φ by Cnst(φ). Thus Cnst($\top$) is the set of all atoms of $\mathscr{B}_{\mathscr{L}}$.[5] From the definition of f: $\mathscr{A}_{\mathrm{c}} \mapsto \mathscr{B}_{\mathscr{L}}$ it immediately follows that f: $\wp(\mathrm{voc}(\mathscr{L})) \mapsto \mathrm{Cnst}(\top)$ is a bijection. It is one-one, since if $C_i \neq C_j$ then $f(C_i) \neq f(C_j)$, and it is onto as for all constituents α there is a C such that $f(C) = \alpha$. The definition of a Lindenbaum algebra implies that for all $\varphi \in \mathrm{Prop}(\mathscr{L})$ the following holds: $\varphi \equiv \alpha_1 \vee \ldots \vee \alpha_n$, $\alpha_i \in \mathrm{Cnst}(\varphi)$. Consequently, F: $\wp(\wp(\mathrm{voc}(\mathscr{L}))) \mapsto \mathscr{B}_{\mathscr{L}}$ defined by $F(X) := f(C_1) \vee \ldots \vee f(C_n)$, $C_i \in X$ is a bijection ($|\mathscr{B}_{\mathscr{L}}| = 2^n$).

If $\mathscr{A}$ and $\mathscr{B}$ are Boolean algebras, then a function g: $\mathscr{A} \to \mathscr{B}$ is a *homomorphism* if:[6]

(1) 1. $g(x \wedge_A y) = g(x) \wedge_B g(y)$ (or $g(x \vee_A y) = g(x) \vee_B g(y)$).
2. $g(\neg_A x) = \neg_B g(x)$

Evidently, f: $\mathscr{A}_c \mapsto \mathscr{B}_{\mathscr{L}}$ is not a homomorphism. For instance, the f-image of the complement of any $B \in \wp(\mathrm{voc}(\mathscr{L}))$ remains a constituent in $\mathscr{B}_{\mathscr{L}}$ whereas the negation of the f-image of B is not a constituent in $\mathscr{B}_{\mathscr{L}}$: $f(B^{\mathrm{c}}) \neq \neg f(B)$).

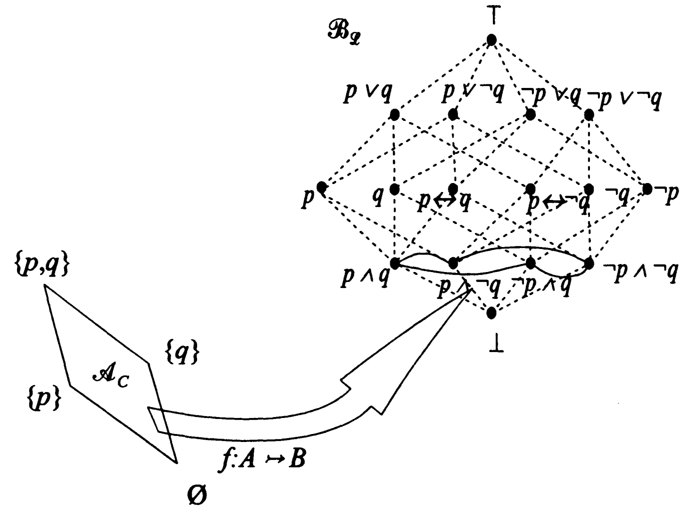


Fig. 2. The mapping f:$\mathscr{A}_c \mapsto \mathscr{B}_{\mathscr{L}}$ for the case $\mathscr{L}[p,q]$

Next, I identify the logical strength of a proposition by its height in the Lindenbaum algebra. First, let me abbreviate the conjunction of $\psi \vDash \varphi$ and $\psi \neq \varphi$ by $\psi \vDash_{\neq} \varphi$. If $\psi \vDash_{\neq} \varphi$ and there is no ξ such that $\psi \vDash_{\neq} \xi \vDash_{\neq} \varphi$, then we say that φ *covers* ψ (see e.g. Birkhof (1948, p. 4)). The line segments in a Hasse diagram represent this covering relation. Further, a *chain* is a set, totally ordered by $\vDash_{\neq}$ and the *height* of φ in $\mathscr{B}_{\mathscr{L}}$, $h(\varphi)$, is the least upper bound of the lengths n of the $\mathscr{L}$-chains,

$$\perp \equiv \varphi_0 \vDash_{\neq} \varphi_1 \vDash_{\neq} ,\ldots ,\vDash_{\neq} \varphi_n \equiv \varphi. \quad (*)$$

Now we can identify the *logical strength of* φ in $\mathscr{L}$ with its height in $\mathscr{B}_{\mathscr{L}}$ (note that ψ is stronger than φ iff $h(\psi) \leq h(\varphi)$). Thus, h will be a variable indicating logical strength.

Observation 6.1: For any finite $\mathscr{B}_{\mathscr{L}}$: $h(\varphi) = |\mathrm{Cnst}(\varphi)|$.
Proof: Let $h(\varphi)$ be the lub of (*)-string lengths. In those strings of maximal length, all $\vDash_{\neq}$ have to be the covering relation, and if φ_{h+1} covers φ_h, then $|\mathrm{Cnst}(\varphi_{h+1}) - \mathrm{Cnst}(\varphi_h)| = 1$. Hence, $h(\varphi_{(n)}) = n = |\mathrm{Cnst}(\varphi)|$. ⊠

In the following subsections it is assumed that the propositional language $\mathscr{L}$ is finite and definite. Thus, the strongest empirical truth τ is complete (τ is a constituent); and the language use finitely many propositional atoms. Moreover, I assume that the empirical truth affirms all the atomic propositions; i.e. $\tau := \wedge p_i$, $p_i \in \mathrm{voc}(\mathscr{L})$. Consequently, application of the definition brings about some rewriting of the language.[7]

6.2.2. The Likeness Refinement

In this subsection I present my comparative truthlikeness order $\preceq_h$ based on the f-mapping of the constituent algebra $\mathscr{A}_c$ into the Lindenbaum algebra $\mathscr{B}_{\mathscr{L}}$ of the preceding subsection. Let us equate the mapping $f^{-1}(\alpha)$ with $\mathrm{pl}(\alpha)$. Thus $\mathrm{pl}\colon \mathrm{Cnst}(\top) \mapsto \wp(\mathrm{voc}(\mathscr{L}))$ gives, for all constituents α, the set of atomic propositions that occur in α without negation. For example, $\mathrm{pl}(\neg p \wedge q \wedge r) = \{q, r\}$ ($\mathrm{pl}(\alpha)$: positive literals of α).

DEFINITION 6.3: Suppose $\psi, \varphi \in \mathrm{Prop}(\mathscr{L})$. Then ψ is *at least as truthlike as* φ iff $\exists\delta\colon \mathrm{Cnst}(\psi) \mapsto \mathrm{Cnst}(\varphi)$ satisfying the following conditions

1. δ is a (non-empty) bijection; $|\mathrm{Cnst}(\psi)| = |\mathrm{Cnst}(\varphi)| = h$ $(h \geq 1)$
2. $\forall\alpha \in \mathrm{Cnst}(\psi)$: $\mathrm{pl}(\alpha) \supseteq \mathrm{pl}(\delta(\alpha))$

Notation: $\psi \preceq_h \varphi$

Obviously, if $\psi \preceq_h \varphi$, then $|\mathrm{Cnst}(\psi)| = |\mathrm{Cnst}(\varphi)|$ and, in $\mathscr{B}_{\mathscr{L}}$, $h(\psi) = h(\varphi)$. The following shows that regarding $\mathscr{B}_{\mathscr{L}}$-constituents, definition 6.3 (the "counting method") equals the symmetric difference proposal for elements of $\mathscr{A}_c$, if the truth is complete.

PROPOSITION 6.2: Suppose $\alpha, \beta, \tau \in \mathrm{Cnst}(\mathscr{L})$ and $\mathrm{pl}(\tau) = \mathrm{voc}(\mathscr{L})$; then $\beta \preceq_1 \alpha \Leftrightarrow \mathrm{pl}(\beta)\,\Delta\,\mathrm{pl}(\tau) \subseteq \mathrm{pl}(\alpha)\,\Delta\,\mathrm{pl}(\tau)$.

Proof: $\beta \preceq_1 \alpha$ is equivalent to $\mathrm{pl}(\beta) \supseteq \mathrm{pl}(\alpha)$, which equals $[\mathrm{pl}(\beta)]^c \subseteq [\mathrm{pl}(\alpha)]^c$. This is the same as $\mathrm{pl}(\tau) - \mathrm{pl}(\beta) \subseteq \mathrm{pl}(\tau) - \mathrm{pl}(\alpha)$ and since $\mathrm{pl}(\tau) = \mathrm{voc}(\mathscr{L})$, the last inclusion equals $[(\mathrm{pl}(\beta) - \mathrm{pl}(\tau)) \cup (\mathrm{pl}(\tau) - \mathrm{pl}(\beta))] \subseteq [(\mathrm{pl}(\alpha) - \mathrm{pl}(\tau)) \cup (\mathrm{pl}(\tau) - \mathrm{pl}(\alpha))]$ which may be abbreviated by $\mathrm{pl}(\beta)\,\Delta\,\mathrm{pl}(\tau) \subseteq \mathrm{pl}(\alpha)\,\Delta\,\mathrm{pl}(\tau)$. ⊠

Observation 6.3: Let $\mathcal{B}_{\mathcal{L}}$ be a Lindenbaum Algebra; then $\preceq_h$ partially orders the set $\{\varphi_h \in \mathcal{B}_{\mathcal{L}} \mid h(\varphi) = h\}$.

Proof: We must prove that $\preceq_h$ is reflexive, transitive and antisymmetric. First, the $\preceq_h$-order is reflexive by definition. Second, let $\psi \preceq_h \varphi \preceq_h \chi$ and δ is the bijection between ψ and φ, and δ' relates φ and χ; then $\delta' \circ \delta$ defines the required bijection between ψ and χ, since for all α in Cnst(ψ): $\mathrm{pl}(\alpha) \supseteq \mathrm{pl}(\delta(\alpha)) \supseteq \mathrm{pl}(\delta'(\delta(\alpha)) = \mathrm{pl}((\delta' \circ \delta)(\alpha))$; therefore $\preceq_h$ is transitive. As to antisymmetry, suppose that $\psi \preceq_h \varphi$, $\varphi \preceq_h \psi$; and δ: Cnst(ψ) $\rightarrowtail$ Cnst(φ) δ': Cnst(φ) $\rightarrowtail$ Cnst(ψ) are bijections. Then $\delta' = \delta^{-1}$. For suppose the contrary; as Cnst(φ) is finite, and δ, δ' are bijections, $\exists \alpha \in \mathrm{Cnst}(\psi)$: $\mathrm{pl}(\alpha) \supset \mathrm{pl}((\delta' \circ \ldots \circ \delta)(\alpha))$ in which $|(\delta' \circ \ldots \circ \delta)| \leq 2h$; contradiction, thus $\delta' = \delta^{-1}$, and δ must be the identity relation such that $\varphi = \psi$; and $\preceq_h$ is antisymmetric. ⊠

Figure 2 illustrates the truthlikeness order $\preceq_h$ in the case of two atomic propositions. If two elements are connected by an arrow, the one at the left is more truthlike than the one on the right. The diagram shows that the order connects only elements of the same logical strength. We saw that $\preceq_h$ partially orders the propositions of the same logical strength; however, $\langle \mathrm{Prop}(\mathcal{L}), \cup_h \preceq_h \rangle$ is not a lattice; $\cup_h \preceq_h$ does not relate propositions of different strengths.

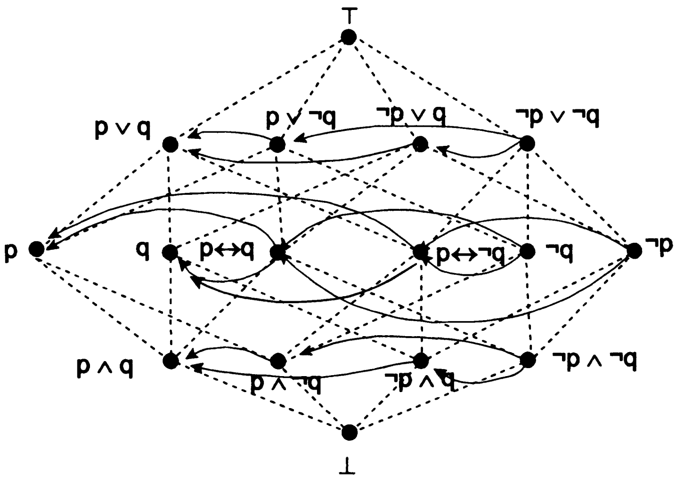


Fig. 3. $\preceq_h$ in $\mathcal{L}[p,q]$ (horizontal arrows)

Let us examine the properties of the likeness relation $\preceq_h$. Truth-value dependence is an important property of approach-to-the-truth definitions. Recall that I called order $\leq_\tau$ *truth-value dependent* if for all φ and ψ in Prop($\mathcal{L}$) obtains that if $\tau \models \psi$ and $\varphi \models \neg\tau$, then $\varphi \nleq_\tau \psi$.

Observation 6.4: The $\preceq_h$-order is truth-value dependent.

Proof: Let τ be the complete truth of $\mathscr{L}$, let $\tau \vDash \psi$ and $\varphi \vDash \neg\tau$ and let $h(\psi) = h(\varphi)$. The definition of $\preceq_h$ immediately implies that $\varphi \npreceq_h \psi$, since $\tau \in \mathrm{Cnst}(\psi)$ and $\tau \notin \mathrm{Cnst}(\varphi)$, and for all elements $\alpha \neq \tau$ of $\mathrm{Cnst}(\top)$ holds $\mathrm{pl}(\tau) \supset \mathrm{pl}(\alpha)$. ⊠

Further, in Chapter 1, I stressed the importance of the language dynamic behaviour of approach-to-the-truth definitions. Let, for all $\mathscr{L}'$ with $\mathscr{L} \subseteq \mathscr{L}'$, a definition establish the orders $\leq_\tau$ and $\leq_{\tau'}$ of $\mathrm{Prop}(\mathscr{L})$ and $\mathrm{Prop}(\mathscr{L}')$, respectively. Let $\tau' \in \mathrm{Prop}(\mathscr{L}')$ be the complete truth in $\mathscr{L}'$, and let $\varphi' \in \mathrm{Prop}(\mathscr{L}')$ be equivalent to $\varphi \in \mathrm{Prop}(\mathscr{L})$. The definition establishing the $\leq_\tau$ and $\leq_{\tau'}$ relations is weakly (strongly) context independent iff

$$\forall\psi, \varphi \in \mathrm{Prop}(\mathscr{L}):\ \psi <_\tau \varphi \Rightarrow \neg(\varphi' <_{\tau'} \psi') \qquad (\textit{weakly})$$
$$\forall\psi, \varphi \in \mathrm{Prop}(\mathscr{L}):\ \psi \leq_\tau \varphi \Rightarrow \psi' \leq_{\tau'} \varphi' \qquad (\textit{strongly})$$

PROPOSITION 6.5: The $\preceq_h$-definition is *strongly context independent.*

Proof: Suppose $\psi \preceq_h \varphi$ for some finite propositional $\mathscr{L}$. Then, there is a bijection δ: $\mathrm{Cnst}(\psi) \rightarrowtail \mathrm{Cnst}(\varphi)$ such that for all $\alpha \in \mathrm{Cnst}(\psi)$: $\mathrm{pl}(\alpha) \supseteq \mathrm{pl}(\delta(\alpha))$ (*). Let $\mathscr{L}' \supseteq \mathscr{L}$ be a finite propositional language, and consider language $\mathscr{L}''$ with $\mathrm{voc}(\mathscr{L}'')$ $= \mathrm{voc}(\mathscr{L}') - \mathrm{voc}(\mathscr{L})$. $\mathrm{Cnst}(\top'')$ consist of 2^n constituents β_j if $|\mathrm{voc}(\mathscr{L}'')| = n$. Obviously, for any $\alpha \in \mathrm{Cnst}(\top)$ and $\beta \in \mathrm{Cnst}(\top'')$, $\alpha \wedge \beta$ is a constituent of $\mathscr{L}'$. Then, $\psi' = \vee_i \vee_j(\alpha_i \wedge \beta_j)$ is the translation of any $\psi = \vee_i\alpha_i \in \mathrm{Prop}(\mathscr{L})$. From (*) it follows that for all $\alpha_i \in \mathrm{Cnst}(\psi)$ $\mathrm{pl}(\alpha_i \wedge \beta_j) \supset \mathrm{pl}(\delta'(\alpha_i \wedge \beta_j))$ obtains, if $\delta'(\alpha_i \wedge \beta_j)$ $:= \delta(\alpha_i) \wedge \beta_j$, Consequently, $\psi' \preceq_h \varphi'$ ⊠

6.2.3. Merging Δ-Content and Δ-Likeness

In this subsection, we see that the Δ-definition and our truthlikeness refinement are compatible; I introduce the provisional refined Δ-verisimilitude definition, and examine its properties. First, let me specify the notion of 'compatibility between two orderings'.

DEFINITION 6.4: The strict partial ordering relations $<$ and $\prec$ on A are compatible iff for all $\psi, \varphi \in A$: $\psi < \varphi \Rightarrow \neg(\varphi \prec \psi)$.

Reflexive partial ordering relations are compatible if the preceding definition obtains for the irreflexive companions of these relations. Evidently, the compatibility of ordering relations is reflexive and symmetric; however, it fails to be transitive. If $\alpha \asymp \beta$, $\alpha \prec \beta$, and $\beta \sqsubset \alpha$, then $\prec$ and $<$ are compatible, just as $<$ and $\sqsubset$; but $\prec$ and $\sqsubset$ are incompatible.

PROPOSITION 6.6: Let τ be complete; then the $\leq^\Delta$-relation (p. 49) is compatible with the $\preceq_h$-relation.

Proof: We have to prove that $\psi \prec \varphi \Rightarrow \neg(\varphi <^{\Delta} \psi)$. Let $\psi \prec \varphi$; it follows that $\varphi \neq \psi$. Now we have to consider two cases; 1. the truth-values of ψ and φ are equal, $tv(\psi) = tv(\varphi)$, and 2. they are unequal $tv(\psi) \neq tv(\varphi)$. 1. assume that $tv(\psi) = tv(\varphi)$; then, $h(\psi) = h(\varphi)$, and since $\varphi \nvDash \psi$ and $\psi \nvDash \varphi$ by proposition 2.3 (p. 53) it follows that $\neg(\varphi <^{\Delta} \psi)$. 2. let $tv(\psi) \neq tv(\varphi)$; then observation 6.4 (p. 198) $\psi \prec \varphi$ entails ψ is true and φ is false. Then, proposition 2.3 again shows that ψ and φ are comparable by $<^{\Delta}$ only if $\psi \vDash \varphi \vee \tau$, which implies $\psi <^{\Delta} \varphi$; and again $\neg(\varphi <^{\Delta} \psi)$. ⊠

The transitive closure is used to merge the $\leq^{\Delta}$-order and the $\preceq_h$-order on $\mathcal{B}_{\mathcal{L}}$. Let ρ be a (binary) relation on A. Then the transitive closure of ρ, $\overline{\rho}$, is defined by: $a\ \overline{\rho}\ b$ iff $\exists n \in \mathbb{N}$: $\exists a_0, a_1, \ldots, a_n \in A$ such that $a = a_0\ \rho\ a_1\ \rho\ \ldots\ \rho\ b = a_n$.[8] This closure enables me to formulate my first approach-to-the-truth proposal. Note the mnemonic representation of the transitive closure. The same scheme will be used in the next sections .

DEFINITION 6.5: ψ is *at least as refined* Δ*-verisimilar as* φ iff

$$\psi\ \overline{\leq^{\Delta} \text{ or } \preceq_h}\ \varphi$$

Notation: $\psi \leq^{\Delta}_{\preceq} \varphi$

In other words, ψ is at least as close to the truth as φ iff there is a chain $\psi = \chi_1\ \rho\ \chi_2\ \rho \cdots \rho\ \chi_{n-1}\ \rho\ \chi_n = \varphi$ in which ρ is the "$\leq^{\Delta}$ or $\preceq_h$"-relation.

Now that I have introduced the definition, let us turn to its consequences. To begin with, we observe that the $\leq^{\Delta}_{\preceq}$- proposal is *strictly stronger* than the $\leq^{\Delta}$-order. For all ψ and φ if $\psi \leq^{\Delta} \varphi$, then $\psi \leq^{\Delta}_{\preceq} \varphi$; this implication does not obtain in the opposite direction. If $tv(\psi) = tv(\varphi)$, and $\psi \neq \varphi$, then $\psi \preceq_h \varphi$ implies $\psi \nleq^{\Delta} \varphi$. Second, we saw that the $\leq^{\Delta}$-order of Prop($\mathcal{L}$) is a lattice (prop. 2.8, p. 54). The refined Δ-verisimilitude order is not a lattice; e.g. p and q do not have a lub in $\langle$Prop($\mathcal{L}[p,q]$), $\leq^{\Delta}_{\preceq}\rangle$. Yet, for all $\varphi \in$ Prop($\mathcal{L}$): $\tau \leq^{\Delta}_{\preceq} \varphi \leq^{\Delta}_{\preceq} \neg\tau$.

PROPOSITION 6.7: The $\leq^{\Delta}_{\preceq}$-order is truth-value dependent.

Proof: In Section 2.5 we saw that the Δ-definition is truth-value dependent. Let $\tau \vDash \psi$ and $\varphi \vDash \neg\tau$; then $\exists \mathfrak{M} \in$ Mod(τ): $\mathfrak{M} \vDash \psi$ and $\mathfrak{M} \nvDash \varphi$ therefore $(\psi \leftrightarrow \tau) \nvDash (\varphi \leftrightarrow \tau)$. Second, the $\preceq_h$-definition is truth-value dependent (obs. 6.3, p.198). As $\leq^{\Delta}$ and $\preceq_h$ both are truth-value dependent, the transitive closure of their disjunctive must have the same property. ⊠

PROPOSITION 6.8: $\leq^{\Delta}_{\preceq}$ is weakly context independent.

Proof: Let for arbitrary $\psi, \varphi \in$ Prop($\mathcal{L}$), $\psi \leq^{\Delta}_{\preceq} \varphi$, and let ψ' and $\varphi' \in$ Prop($\mathcal{L}'$) be the translations of ψ and φ in $\mathcal{L}' \supset \mathcal{L}$. Now, suppose that $\varphi' \leq^{\Delta}_{\preceq} \psi'$; as $\leq^{\Delta}_{\preceq}$ is truth-value dependent, this implies that $tv(\psi) = tv(\varphi)$. Since the $\preceq_h$-definition is strongly context independent, and the $\leq^{\Delta}$-transitions reduce to the $\vDash$-relation for ψ and φ, with the same truth-value, $\leq^{\Delta}_{\preceq}$ is *strongly context independent.* Consequently, $\psi \leq^{\Delta}_{\preceq} \varphi$ implies $\psi' \leq^{\Delta}_{\preceq} \varphi'$, which contradicts the assumption $\varphi' \leq^{\Delta}_{\preceq} \psi'$; therefore $\varphi' \not\leq^{\Delta}_{\preceq} \psi'$. ⊠

The next question reads of course, whether the $\leq^{\Delta}_{\preceq}$-order is also *strongly* context independent. In Chapter 2 (p. 57) I concluded that the $\leq^{\Delta}$-order fails to be strongly context independent, as $p \leq^{\Delta}_{p} \neg p$ but the $\leq^{\Delta}_{p\wedge q}$ relation fails to order p and $\neg p$. This example does not establish the same result for the $\leq^{\Delta}_{\preceq}$-order since $p \prec_h \neg p$ in all languages containing p. An analogous switch from $\mathscr{L}[p,q]$ to $\mathscr{L}[p,q,r]$ reveals, however, that the example is sheer coincidence. In $\mathscr{L}[p,q]$, $p \wedge q \leq^{\Delta}_{p\wedge q} \neg p \vee \neg q$, whereas in $\mathscr{L}[p,q,r]$, $p \wedge q \not\leq^{\Delta}_{p \wedge q \wedge r} \neg p \vee \neg q$, and the $\prec_h$ relation does not order $\neg p \vee \neg q$ and $p \wedge q$ since $h(\neg p \vee \neg q) \neq h(p \wedge q)$. Hence, the $\leq^{\Delta}_{\preceq}$-order is not strongly context independent.

Figure 3 pictures the genealogy of the $\leq^{\Delta}_{\preceq}$-order for $\mathscr{L}[p,q]$ with the truth $\tau := p \wedge q$. The diagram in the upper right corner illustrates that, as mentioned already, $\psi \preceq_h \varphi$ implies $h(\psi) = h(\varphi)$. Further comparison of the diagram in the upper left corner with the one in the upper right corner demonstrates that if $tv(\psi) = tv(\varphi)$, then $\psi \preceq_h \varphi$ implies that ψ and φ are not related by the $\leq^{\Delta}$-ordering; and if $tv(\psi) >$ tv(φ), then $\psi \leq^{\Delta} \varphi$ implies $\psi \preceq_h \varphi$. Moreover, the upper and lower layer of the $\leq^{\Delta}_{\preceq}$-order, in the lower left corner, display a remarkable congruence. In the next section I shall explain this congruence. It is not a coincidence but a structural phenomenon for (finite) propositional languages, even if the truth is incomplete; although incompleteness of the truth results in more than two layers.

Besides the genealogy of the $\leq^{\Delta}_{\preceq}$-order on Prop($\mathscr{L}[p,q]$), Figure 3 illustrates the architecture of the next two sections. In Section 6.3 I introduce vertical, and in Section 6.4 I define horizontal equivalence classes in Prop($\mathscr{L}$). On the one hand, propositions connected by a vertical line segment in the diagrams of Figure 3 at the left-hand side are members of an equivalence class of Section 6.3; propositions $p \vee q$ and $p \leftrightarrow \neg q$ provide an example. On the other hand, in Section 6.4 the $\preceq_h$-order is strengthened by assuming equal importance of all atomic propositions; therefore a uniform substitution of free atomic propositions yields equally truthlike theories; this strengthened truthlikeness relation is invariant under uniform substitution of atomic propositions. For instance, $p \vee \neg q$ and $q \vee \neg p$ are equally truthlike, and belong to the same (truthlikeness) equivalence class. Although uncomparable, these two propositions are situated alongside in the lower left corner diagram. Finally, in Section 6.4, the "horizontal" and "vertical" partitions (dotted lines) are merged; this merger orders the elements of the resulting partition (lower right corner).

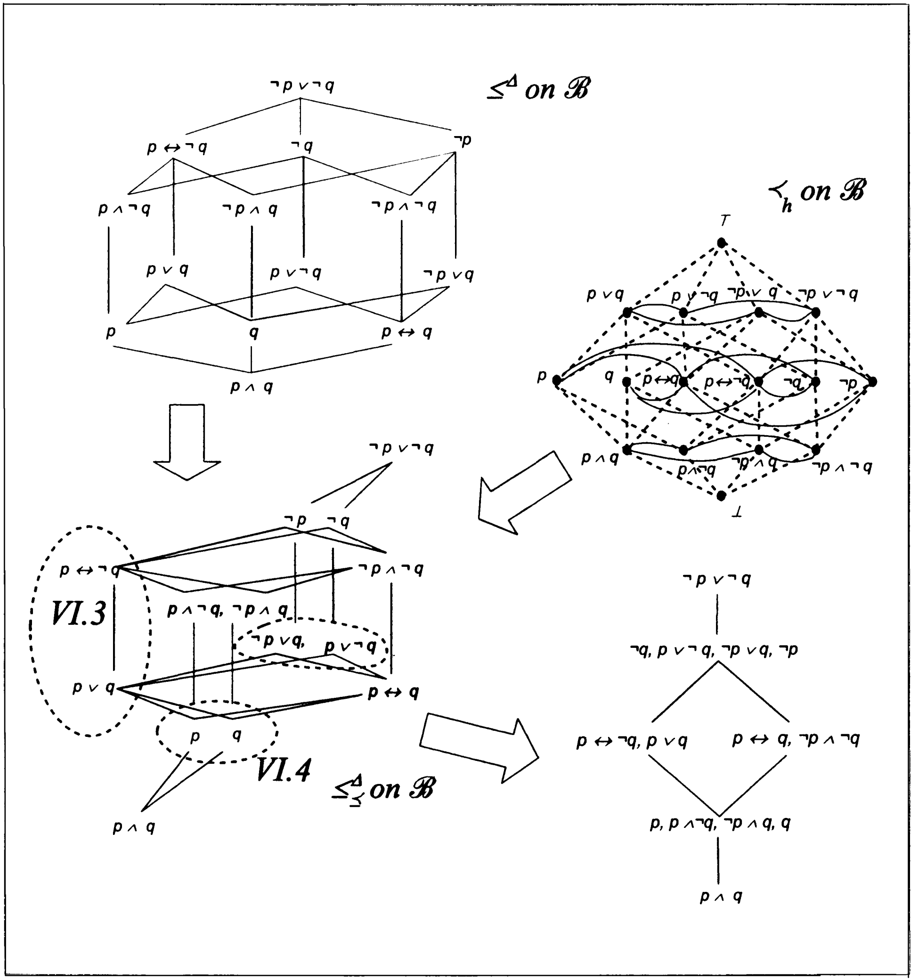


Fig. 4. The genealogy of the $\leq^{\Delta}_{\preceq}$-order on $\mathscr{B}_{\mathscr{L}}[p,q]$

Let me summarize the current section. First, we considered a bijection between the constituent algebra and the atoms of the Lindenbaum algebra of a finite propositional language $\mathscr{L}$. This mapping induced a truthlikeness order on the set of propositions of the same logical strength. Then we used the transitive closure of the "$\leq^{\Delta}$ or $\preceq_h$" to merge the $\leq^{\Delta}$-order and the likeness order. This resulted in a weakly context independent strengthening of the original Δ-proposal, which remained truth-value dependent.

6.3. THE CONSEQUENCE DEFINITION: $\leq^+$

Although the refined Δ-verisimilitude definition has advantages over the original Δ-definition, it still has its drawbacks. The most important disadvantage is that the new proposal inherits the truth-value dependency of the Δ-definition. Therefore I substitute the consequence definition (or +-definition) for the Δ-definition, since, as we showed in Chapter 2 the +-definition is truth-value independent (p. 55). This substitution involves an introduction of an equivalence class on $\mathscr{B}_{\mathscr{L}}$. Let us explain these two assertions.

In Chapters 1–2, we showed that according to Popper's original proposal and the Δ-definition a predominance of truth content is a necessary condition for being more verisimilar. Consequently, ψ is more verisimilar than φ only if $\psi \vDash \varphi \vee \tau$ or $\mathrm{Cn}(\varphi) \cap \mathrm{Cn}(\tau) \subseteq \mathrm{Cn}(\psi)$, which equals the definiens of the +-definition. Observe that the definiens allows ψ to be false and φ to be true. The Δ-definition adds $\varphi \wedge \tau \vDash \psi$ to the +-definiens, if the truth is axiomatizable. This addition renders the Δ-proposal truth-value dependent; φ is true and ψ false entails $\varphi \wedge \tau \nvDash \psi$. Let us turn to the second assertion.

In Chapter 2, we showed that the Δ-definition *partially* orders $\mathscr{B}$, the Lindenbaum algebra of the propositions; two propositions have the same Δ-distance to the truth iff they are equivalent. Our syntactic formulation of the Δ-definition illustrates this point. It reads:

(2) $\psi \leq^{\Delta} \varphi$ *iff* $\varphi \leftrightarrow \tau \vDash \psi \leftrightarrow \tau$

Clearly, the $\leq^{\Delta}$-order is transitive and reflexive, and its antisymmetry; becomes evident by showing that $\psi \leftrightarrow \tau \equiv \varphi \leftrightarrow \tau$ implies $\vDash \psi \rightarrow \varphi$, and $\vDash \varphi \rightarrow \psi$. In contrast to the $\leq^{\Delta}$-order, the $\leq^+$-order of $\mathscr{B}$, is a *preorder*. Its transitivity and reflexivity are obvious, but the $\leq^+$-order lacks antisymmetry; the conjunction of $\psi \vDash \varphi \vee \tau$ and $\varphi \vDash \psi \vee \tau$ does not imply $\varphi \equiv \psi$ (e.g. $\tau := p \wedge q$, $\psi := p \leftrightarrow q$ and $\varphi := \neg p \wedge \neg q$). Furthermore, a preorder $\leq$ partially orders the equivalence classes $[\varphi]$ defined by $\varphi' \in [\varphi]$ iff $\varphi' \leq \varphi$ and $\varphi \leq \varphi'$.

Let me summarize the two subsections that are to come. The algebraic prerequisites will be explained in subsection 6.3.1. I consider the ideal generated by the truth τ, I_τ, which introduces a congruence relation on $\mathscr{B}$, the equivalence classes of which form the quotient algebra $\mathscr{B}/I_\tau$. I prove in subsection 6.3.2. that $\mathscr{B}/I_\tau$ is isomorphic with the $\leq^+$-order of $\mathscr{B}$; it is possible to define a homomorphism between the $\leq^+$-order and $\mathscr{B}$. Besides truth-value independence, this is another reason for substituting the $\leq^+$-order for the Δ-definition. We need this homomorphism result to prove the theorems in Sections 6.4–6.5. Additionally, I reconsider the relation between the Cn- and Δ-definition, using algebraic terms, and discuss the language dynamics of the Cn-proposal.

6.3.1. Algebraic Prerequisites

The first algebraic concept I need is that of an *ideal* in a Boolean algebra. Intuitively, an ideal generated by $u \in \mathcal{B}$ is the set of all elements "smaller than u."

Reminder: An *ideal* of a Boolean algebra $\mathcal{B}$ is a non-empty $I \subset \mathcal{B}$ such that
1. $x \in I$ and $y \in I$ implies $x \vee y \in I$
2. $x \in I$ and $y \in \mathcal{B}$ implies $x \wedge y \in I$

If $I \neq \mathcal{B}$ then I is *proper*; and if $u \in \mathcal{B}$ then $I_u := \{x|x \leq u\}$ is a *principal ideal* generated by u.

If $\mathcal{B}$ is a Lindenbaum algebra, then I_τ equals An(τ) (the dual of an ideal is a filter). The next reminder defines a proper congruence relation. Intuitively, a congruence relation partitions an Boolean algebra into "isomorphic equivalence classes."

Reminder: Relation θ ($\neq$ universal relation) is a (proper) *congruence* on $\mathcal{B}$ iff
1. θ is an equivalence relation on $\mathcal{B}$;
2. If $b\ \theta\ c$, then $(b \wedge d)\ \theta\ (c \wedge d)$ for all d;
3. If $b\ \theta\ c$, then $b'\ \theta\ c'$ (b' and c' are the complements in the algebra).

Observation 6.9: If θ is a congruence relation on $\mathcal{B}$, then $b\ \theta\ c$ implies for all d
$b \vee d\ \theta\ c \vee d$

Proof: Suppose $b\ \theta\ c$. Then $\neg b\ \theta\ \neg c$ (from 3.), which implies $\neg b \wedge \neg d\ \theta\ \neg c \wedge \neg d$ (from 2.). Thus $\neg(b \vee d)\ \theta\ \neg(c \vee d)$ implying $b \vee d\ \theta\ c \vee d$. ⊠

The following well-known theorem of the theory of lattices relates ideals and congruences using the symmetric difference. In the words of Stoll (1961, p. 265):

Reminder: If θ is a proper congruence relation on a Boolean algebra $\mathcal{B}$, then $I := \{a \in B|\ a\ \theta\ \bot_{\mathcal{B}}\}$ is a proper ideal of $\mathcal{B}$, and $a\ \theta\ b$ iff $a \Delta b \in I$. Conversely, if I is a proper ideal of $\mathcal{B}$, then the relation θ defined by: $a\ \theta\ b$ iff $a \Delta b \in I$, is a proper congruence relation on $\mathcal{B}$ with $I = \{a \in \mathcal{B}|\ a\ \theta\ \bot_{\mathcal{B}}\}$.

The theorem says that the equivalence: $a\ \theta\ b$ iff $a \Delta b \in I$ defines a bijection between the ideals and congruences of a Boolean algebra. In the context of a Lindenbaum algebra with truth τ, the previous theorem entails that the ideal generated by the truth, the truth-ideal I_τ, corresponds to the truth-congruence $\sim_\tau$ on $\mathcal{B}$;[9]

$$\psi \sim_\tau \varphi := \psi \Delta \varphi \in I_\tau$$

Recall that in a Lindenbaum algebra, $\psi \sim_\tau \varphi :=_{def} (\psi \leftrightarrow \neg\varphi) \vDash \tau)$.

Let us consider the quotient algebra $\mathcal{B}/I$ of all equivalence classes $[x] := \{y|\ x \sim_\tau y\}$ in $\mathcal{B}$. In lattice theory. It is a well-known fact that if I is a proper ideal of a Boolean algebra $\mathcal{B}$, then there is a *homomorphic* mapping from$\mathcal{B}$ to $\mathcal{B}/I$. The converse of this contention also obtains. In a Boolean algebra there is a one-one correspondence between homomorphisms and proper congruence relations, and proper ideals. Referring to the quotient algebra, I use the following notations if τ generates I_τ:

$\mathcal{B}/I := \langle B/I; \wedge_{\mathcal{B}/I}, \vee_{\mathcal{B}/I}, \neg_{\mathcal{B}/I}, \bot_{\mathcal{B}/I}, \top_{\mathcal{B}/I}\rangle$ in which
$B/I := \{[x] \mid x \in \mathcal{B}\}$
$[x] \wedge_{\mathcal{B}/I} [y] := [x \wedge y]$
$[x] \vee_{\mathcal{B}/I} [y] := [x \vee y]$
$\bot_{\mathcal{B}/I} := [\bot] = I$
$\top_{\mathcal{B}/I} := [\top] = \{y \mid \neg y \in I\}$
$\neg_{\mathcal{B}/I}[x] := [\neg y]$

As usual, $[x] \leq_{\mathcal{B}/I} [y] :=_{def} [x] \wedge_{\mathcal{B}/I} [y] = [x]$ defines the partial order on $\mathcal{B}/I$. In what follows, I want to describe the relation between the truth-ideal I_τ, the Boolean algebra $\mathcal{B}$ and their quotient algebra $\mathcal{B}/I$. As a preliminary step, I sketch the extension of an ideal into a Boolean algebra. Next, I discuss the isomorphism between the Boolean extensions of An($\neg\tau$)—i.e. $\mathcal{I}_{\neg\tau}$—and the quotient algebra $\mathcal{B}/I$, using the Boolean algebra version of the Fundamental Homomorphism Theorem.[10]

The next definitions extend a proper ideal I_τ of $\mathcal{B}$ generated by τ into a Boolean algebra $\mathcal{I}_\tau$. $\mathcal{I}_\tau$ is a Boolean extension of I_τ, iff $\mathcal{I}_\tau := \langle I_\tau; \wedge_{\mathcal{I}} \vee_{\mathcal{I}}, \neg_{\mathcal{I}}, \bot_{\mathcal{I}}, \top_{\mathcal{I}}\rangle$ and $\wedge_{\mathcal{I}} := \wedge_{\mathcal{B}}\restriction_{I_\tau}$; $\vee_{\mathcal{I}} := \vee_{\mathcal{B}}\restriction_{I_\tau}$; $\bot_{\mathcal{I}} := \bot_{\mathcal{B}}$; $\top_{\mathcal{I}} := \tau$; and $\neg_{\mathcal{I}}\ \varphi := \tau \wedge \neg\varphi$. Since I_τ is an ideal, it is closed under lub and glb of $\mathcal{B}$, and $\mathcal{I}_\tau$ is a distributive lattice. Moreover, the definition of the complement in $\mathcal{I}_\tau$ implies $\varphi \wedge\neg_{\mathcal{I}}\ \varphi = \bot$, and $\varphi \vee\neg_{\mathcal{I}}\ \varphi = \tau$. Hence, $\mathcal{I}_\tau$ is a Boolean algebra. Since only the complement in $\mathcal{I}$ deviates from the $\mathcal{B}$-complement, I shall omit subscripts for the other operations. As to the relation between $\mathcal{B}$, $\mathcal{B}/I_\tau$ and I_τ I consider the Boolean algebra $\mathcal{I}_{\neg\tau}$. The mapping $f(\varphi) := \varphi \wedge \neg\tau$ defines a homomorphism from $\mathcal{B}$ onto $\mathcal{I}_{\neg\tau}$, since $f(\varphi \wedge \psi) = \varphi \wedge \psi \wedge \neg\tau = f(\varphi) \wedge f(\psi)$, and $f(\neg\varphi) = \neg\varphi \wedge \neg\tau = \neg_{\mathcal{I}} f(\varphi)$ ($\neg_{\mathcal{I}} f(\varphi) \wedge f(\varphi) = \bot$, and $\neg_{\mathcal{I}} f(\varphi) \vee f(\varphi) = \neg\tau$). Without spelling out the details, the Boolean algebra version of the Fundamental Homomorphism Theorem implies that the equivalence $\sim_\tau$ is a congruence, and $\mathcal{B}/I$ is isomorphic with $\mathcal{I}_{\neg\tau}$, the same obtains, mutatis mutandis, for the homomorphism $g(\varphi) := \varphi \vee \tau$ and extension of Cn(τ) into the Boolean algebra $\mathcal{F}_\tau$[11]. In short, $\mathcal{I}_{\neg\tau}$ and $\mathcal{F}_\tau$, the Boolean extensions of An($\neg\tau$), Cn(τ) are isomorphic with $\mathcal{B}/I$.

Now that I have delineated $\mathcal{I}_\tau$, I sketch the relation between $\mathcal{B}/I$ and I_τ and demonstrate that $\mathcal{B}$ is isomorphic with the *Cartesian product* of $\mathcal{B}/I$ and $\mathcal{I}_\tau$:[12]

$$\mathcal{B} \cong \mathcal{B}/I \times \mathcal{I}_\tau.$$

As $\mathcal{B}/I_\tau$ is isomorphic with $\mathcal{I}_{\neg\tau}$, it suffices to show that $\mathcal{B}$ is isomorphic with the Cartesian product of $\mathcal{I}_{\neg\tau}$ and $\mathcal{I}_\tau$. Let α_1 and $\beta_1 \in \mathcal{I}_\tau$, and α_2 and $\beta_2 \in \mathcal{I}_{\neg\tau}$. Then, the lub of (α_1, α_2) and (β_1, β_2), and the complement in $\mathcal{I}_\tau \times \mathcal{I}_{\neg\tau}$ are respectively,

$$(\alpha_1, \alpha_2) \vee_{\mathcal{I}_\tau \times \mathcal{I}_{\neg\tau}} (\beta_1, \beta_2) :=_{def} (\alpha_1 \vee_{\mathcal{I}_\tau} \beta_1, \alpha_2 \vee_{\mathcal{I}_{\neg\tau}} \beta_2)$$
$$(\alpha_1, \alpha_2) \wedge_{\mathcal{I}_\tau \times \mathcal{I}_{\neg\tau}} (\beta_1, \beta_2) :=_{def} (\alpha_1 \wedge_{\mathcal{I}_\tau} \beta_1, \alpha_2 \wedge_{\mathcal{I}_{\neg\tau}} \beta_2) \text{ and}$$
$$\neg_{(\mathcal{I}_\tau \times \mathcal{I}_{\neg\tau})} (\alpha_1, \alpha_2) :=_{def} (\neg_{\mathcal{I}_\tau}\alpha_1, \neg_{\mathcal{I}_{\neg\tau}}\alpha_2).$$

As usual $(\alpha_1, \alpha_2) \leq_{\mathcal{I}_\tau \times \mathcal{I}_{\neg\tau}} (\beta_1, \beta_2)$ iff $\alpha_i \leq \beta_i$ for $i = 1, 2$ defines the order of $\mathcal{I}_\tau \times \mathcal{I}_{\neg\tau}$.

PROPOSITION 6.10: The mapping f: $\mathcal{I}_\tau \times \mathcal{I}_{\neg\tau} \rightarrow \mathcal{B}$ defined by $f(\alpha_1, \alpha_2) := \alpha_1 \vee_{\mathcal{B}} \alpha_2$ is an isomorphism.

Proof: We must prove that 1. f is a homomorphism from $\mathcal{I}_\tau \times \mathcal{I}_{\neg\tau}$ onto $\mathcal{B}$ (see (1), p. 196) and 2. f is 1-1.[13] 1. $f[(\alpha_1, \alpha_2) \vee_{\mathcal{I}_\tau \times \mathcal{I}_{\neg\tau}} (\beta_1, \beta_2)]$ = (*lub definition in* $I_\tau \times I_{\neg\tau}$) = $f[(\alpha_1 \vee_{\mathcal{I}} \beta_1), (\alpha_2 \vee_{\mathcal{I}_{\neg\tau}} \beta_2)]$ = (*definition f-mapping*) = $(\alpha_1 \vee_{\mathcal{I}} \beta_1) \vee_{\mathcal{B}} (\alpha_2 \vee_{\mathcal{I}_{\neg\tau}} \beta_2)$ = ($\vee_{\mathcal{I}} = \vee_{\mathcal{B}} = \vee_{\mathcal{I}_{\neg\tau}}$) = $(\alpha_1 \vee_{\mathcal{B}} \alpha_2) \vee_{\mathcal{B}} (\beta_1 \vee_{\mathcal{B}} \beta_2)$ = (*f-mapping definition*) $= f(\alpha_1, \alpha_2) \vee_{\mathcal{B}} f(\beta_1, \beta_2)$. Similar reasoning gives $f[(\alpha_1, \alpha_2) \wedge_{\mathcal{I}_\tau \times \mathcal{I}_{\neg\tau}} (\beta_1, \beta_2)] = f(\alpha_1, \alpha_2) \wedge_{\mathcal{B}} f(\beta_1, \beta_2)$. 3. $f(\neg_{\mathcal{I}_\tau \times \mathcal{I}_{\neg\tau}}(\alpha_1, \alpha_2))$ = (*negation in* $\mathcal{I}_\tau \times \mathcal{I}_{\neg\tau}$) = $f(\neg_{\mathcal{I}_\tau}\alpha_1, \neg_{\mathcal{I}_{\neg\tau}}\alpha_2)$ = (*f-mapping definition*) = $\neg_{\mathcal{I}_\tau}\alpha_1 \vee_{\mathcal{B}} \neg_{\mathcal{I}_{\neg\tau}}\alpha_2$. By definition we know that $\tau \vee_{\mathcal{B}} \neg\tau = \top$. Hence $(\alpha_1 \vee_{\mathcal{B}} \neg_{\mathcal{I}_\tau}\alpha_1) \vee_{\mathcal{B}} (\alpha_2 \vee_{\mathcal{B}} \neg_{\mathcal{I}_{\neg\tau}}\alpha_2) = \top$, and $(\alpha_1 \vee_{\mathcal{B}} \alpha_2) \vee_{\mathcal{B}} (\neg_{\mathcal{I}_\tau}\alpha_1 \vee_{\mathcal{B}} \neg_{\mathcal{I}_{\neg\tau}}\alpha_2) = \top$. Consequently, $(\neg_{\mathcal{I}_\tau}\alpha_1 \vee_{\mathcal{B}} \neg_{\mathcal{I}_{\neg\tau}}\alpha_2) = \neg_{\mathcal{B}}(\alpha_1 \vee_{\mathcal{B}} \alpha_2)$ = (*definition f-mapping*) = $\neg_{\mathcal{B}} f(\alpha_1, \alpha_2)$. 2. Suppose $\alpha_1 \vee \alpha_2 = \beta_1 \vee \beta_2$ for $\alpha_1, \beta_1 \in \mathcal{I}_\tau$ and $\alpha_2, \beta_2 \in \mathcal{I}_{\neg\tau}$ (†); furthermore, suppose that $\alpha_1 \neq \beta_1$; then $\alpha_1 \Delta \beta_1 \neq \bot$. Further $\mathcal{I}_\tau \cap \mathcal{I}_{\neg\tau} = \bot$ and since $\alpha_1 \Delta \beta_1 \in \mathcal{I}_\tau$, (†) implies $\alpha_1 \vee \alpha_2 \neq \beta_1 \vee \beta_2$ which contradicts the assumption; $\alpha_2 \neq \beta_2$ implies the same result; therefore $\alpha_1 = \beta_1$ and $\alpha_2 = \beta_2$, and f is 1-1 and a Boolean isomorphism. ⊠

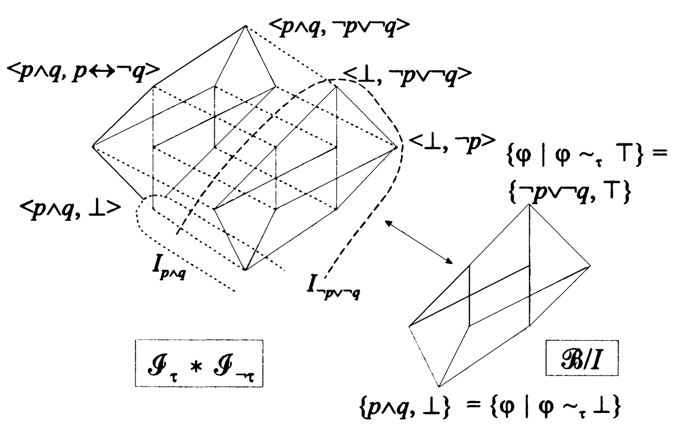


Fig. 5. $\mathcal{B} \cong \mathcal{I}_{p\wedge q} \times \mathcal{I}_{\neg p\vee\neg q}$

As the quotient algebra $\mathcal{B}/I_\tau$ is isomorphic with $\mathcal{I}_{\neg\tau}$ the preceding proposition implies that $\mathcal{B}$ is isomorphic with the Cartesian product of $\mathcal{B}/I_\tau$ and $\mathcal{I}_\tau$. For instance, let $\alpha_1 = \beta_1 := p \wedge q \in \mathcal{I}_\tau$ and let $\alpha_2 := \neg p \wedge \neg q$ and $\beta_2 := \neg p \in \mathcal{I}_{\neg\tau}$. Then, $\alpha_1 \vee \alpha_2 = p \leftrightarrow q \leq_{\mathcal{B}} \beta_1 \vee \beta_2 = \neg p \vee q$. Due to the Cartesian product, the pairs of the left hand side of $\mathcal{I}_\tau \times \mathcal{I}_{\neg\tau}$ all start with $p \wedge q$, the top of $\mathcal{I}_\tau$; the pairs on the right all start with $\perp$, the bottom of $\mathcal{I}_\tau$. The pairs of elements connected with the dotted lines all have the same second element. Figure 4 (p. 206) sketches the isomorphism between $\mathcal{I}_\tau \times \mathcal{I}_{\neg\tau}$ and $\mathcal{B}$ in case of the two atomic propositions example where the truth τ is $p \wedge q$.

Let us summarize the results obtained so far. Suppose $\mathcal{B}$ is a Boolean algebra, and $\mathcal{I}_\tau$ is one of its ideals generated by the atom τ. Then, the congruence relation $\sim_\tau$ defined by $\psi \sim_\tau \varphi$ iff $\psi \,\Delta\, \varphi \in I_\tau$ partitions $\mathcal{B}$. The $\sim$-equivalence classes are the elements of the quotient algebra $\mathcal{B}/I_\tau$, and $\mathcal{B}$ is isomorphic with the Cartesian product of $\mathcal{B}/I_\tau$ and the Boolean extension of I_τ, $\mathcal{I}_\tau$.

6.3.2. The +-definition Again

In this subsection, I first reformulate the +-definition in algebraic terms and discuss its content behaviour. Then, using algebraic means, I reconsider the relation between the +- and Δ-definition; and finally, I discuss the language dynamics of the +-proposal.

My first task is to reformulate the +-definition in algebraic terms. Suppose $\mathcal{B}_{\mathcal{L}}$ is a Lindenbaum algebra of the propositional language $\mathcal{L}$ and I_τ is the ideal generated by the truth τ (τ need not be complete). Further, let me dismiss the tautology and the contradiction as sensible candidates for being verisimilar.[14] Then, the next proposition bridges the gap between the syntactic and the algebraic formulation of the +-definition.

PROPOSITION 6.11: For all $\psi, \varphi \in \mathrm{Prop}(\mathcal{L})$, $[\psi] \leq_{\mathcal{B}/I} [\varphi]$ iff $\psi \vDash \varphi \vee \tau$

Proof: The equivalence follows immediately form the isomorphism between $\mathcal{B}/\mathrm{I}$, $\mathcal{I}_{\neg\tau}$, and $\mathcal{F}_\tau$; however, we proceed as follows: $[\psi] \leq_{\mathcal{B}/\mathrm{I}} [\varphi] \Leftrightarrow f(\psi) \leq_{\mathcal{I}\neg\tau} f(\varphi) \Leftrightarrow \psi \wedge \neg\tau \leq_{\mathcal{I}\neg\tau} \varphi \wedge \neg\tau \Leftrightarrow \psi \wedge \neg\tau \leq_{\mathcal{B}} \varphi \wedge \neg\tau \Leftrightarrow \psi \vDash \varphi \vee \tau$. ⊠

For ψ and φ with $\psi \neq \varphi$, we say ψ is more verisimilar than φ iff $[\psi] \leq_{\mathcal{B}/I}[\varphi]$, i.e. $\psi <^+ \varphi$; and ψ is as verisimilar as φ, $\psi \sim^+ \varphi$, iff $\psi \in [\varphi]$. We already noticed that, although the reflexive formulation of the Δ-definition provides a partial order on $\mathcal{B}$, $\leq^+$ is a preorder on $\mathcal{B}$. It misses the antisymmetry property as the conjunction of $\psi \leq^+ \varphi$ and $\varphi \leq^+ \psi$ does not imply $\psi = \varphi$. As $\leq_{\mathcal{B}/\mathrm{I}}$ is a (weak) partial order on $\mathcal{B}/\mathrm{I}$, proposition 6.11 shows that, unlike the $\leq^\Delta$-order (see fig 5, p. 209), the $\leq^+$-order is homomorphic with the Lindenbaum algebra $\mathcal{B}_{\mathcal{L}}$. In this respect, the +-definition is unique. No other comparative approach-to-the-truth order is homomorphic to the original Lindenbaum algebra. This homomorphism will turn out to be crucial for the proposals of the next sections.

Observe that the +-definition exceeds the Δ-definition in content character. According to the Δ-definition, sometimes truth-value overrules logical strength; if $\psi \equiv \varphi \vee \tau$ and $\varphi \vDash \neg\tau$, $\psi \leq^{\Delta} \varphi$ and $\varphi \nleq^{\Delta} \psi$, and the weaker proposition is closer to the truth. In the same situation, the Cn-proposal counterbalances the extra logical strength of φ against its failure of truth-value; $\psi \leq^{+} \varphi$ and $\varphi \leq^{+} \psi$ obtain both, such that ψ and φ are equally close to the truth. The following preference order is perhaps even more awkward for someone whose intuitions are primarily focused on the atomic propositions instead of the true consequences of a theory. If p is the strongest truth for $\mathscr{L}[p,q]$, $\neg p \wedge \neg q$, $p \vee \neg q$, and $p \leftrightarrow q$ are equally verisimilar, although the first has more false consequences than the second. According to the Cn-proposal, logical strength counterbalances false consequences[15]. Next, I reexamine the contrast between the Δ-proposal and the +-definition using algebraic means.

6.3.3. Removal of the Antecedence Clause

I already mentioned that the +-definition is stronger than the Δ-definition. The two-proposition example nicely illustrates how the removal of *An-clause*, as I called the constraint $\varphi \wedge \tau \vDash \psi$ in Chapter 2, increases the strength of this content proposal. In $\mathscr{L}[p,q]$ where $\tau \equiv p \wedge q$, the +-definition adds six pairs to the $\leq^{\Delta}$-order:

$$p \wedge \neg q \leq^{+} p \vee \neg q \qquad \neg p \wedge \neg q \leq^{+} p \vee \neg q \qquad \neg p \wedge q \leq^{+} p \vee q$$
$$p \wedge \neg q \leq^{+} p \vee q \qquad \neg p \wedge \neg q \leq^{+} \neg p \vee q \qquad \neg p \wedge q \leq^{+} \neg p \vee q$$

For instance, the Δ-definition does not relate $p \wedge \neg q$ and $p \vee \neg q$: $p \wedge \neg q \not\lesseqgtr^{\Delta} p \vee \neg q$, although $\neg p \wedge \neg q \leq^{+} p \vee \neg q$ obtains according to the Cn-proposal. This becomes more of a problem, if the number of atomic propositions increases. Intuitively, $p_1 \wedge \ldots \wedge p_9 \wedge \neg p_{10}$ is closer to $p_1 \wedge \ldots \wedge p_{10}$, than $p_1 \vee \neg p_2 \vee \ldots \vee \neg p_{10}$. As the Cn-proposal verifies this intuition, whereas the Δ-proposal does not, apparently, the first solves some of the problems of the second.

Figure 5 shows how the increase of strength of the Cn-proposal comes about. All pairs of propositions $\langle \varphi \vee \tau, \varphi \rangle$ with $\varphi \vDash \neg\tau$ (the vertical line segments in the left diagram) receive the same verisimilitude; and the $\leq^{\Delta}$-order turns into the Boolean algebra $\mathscr{B}/I_{p\wedge q}$. The two propositional case gives some idea how the $\leq^{\Delta}$-order turns into the $\leq^{+}$-order for $\mathscr{L}[p_1, \ldots p_n]$ with complete truth. The $\leq^{\Delta}$-order consists of two layers, one of true propositions, the other of false ones. The transitions from the $\leq^{\Delta}$-order to the $\leq^{+}$-order puts these two layers together. More generally, if the truth is incomplete, the number of isomorphic layers increases; and again, without the *An-clause*, the *Cn-clause* (or the +-definition) takes these layers together, and renders a partial ordering to the equivalence classes of the truth-ideal related congruence on $\mathscr{B}$.

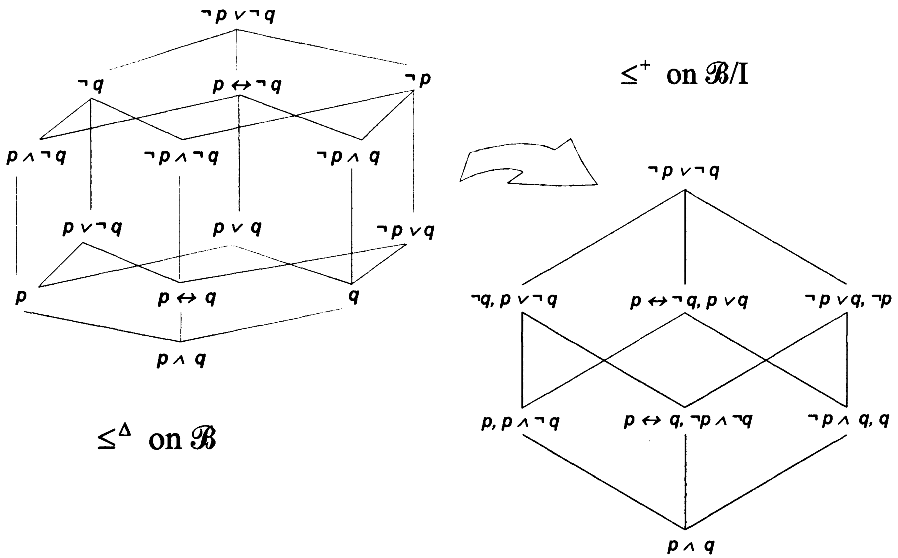


Fig. 6. From the Δ-order to the Pr-order of Prop($\mathscr{L}[p,q]$).

Figure 6 illustrates this point. By presenting the Hasse diagram of the $\leq^{\Delta}$-ordering of Prop($\mathscr{L}[p,q]$), it clearly shows the relation between the Cn-clause and the An-clause of the symmetric difference order, when p is the strongest empirical truth. The first is responsible for the order along the continuous lines; and the second causes the order along the dotted lines.

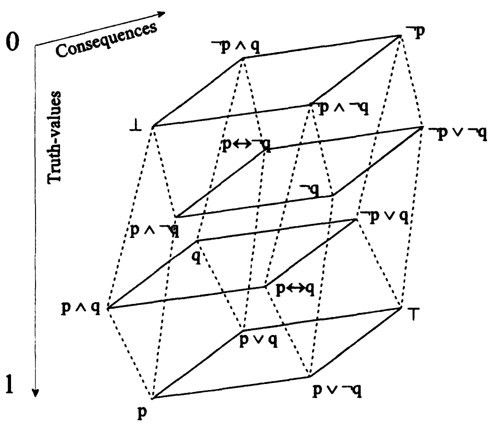


Fig. 7. The Δ-ordering of Prop($\mathscr{L}[p,q]$) with $\tau := p$

The top layer represents the false propositions, and the lowest layer contains the true propositions. Although the consequence relation is also present along the vertical axis, higher propositions imply lower propositions along the dotted lines, its general characteristic is the truth-value distribution over the various layers. According to the consequence definition, the propositions connected by the dotted lines form one equivalence class. The likeness refinements are diagonals parallel to $p \vee q$ and $p \vee \neg q$ in the planes defined by the continuous lines.[16]

I end my comparison of the two content proposals with the next proposition. We already saw that for all φ and ψ of $\mathcal{B}$, $\psi \leq^{\Delta} \varphi$ implies $\psi \leq^{+} \varphi$. Generally, the inverse of this implication does not hold, but for some particular φ' it does. In the preceding subsection we saw that the $\sim_{\tau}$-relation defines a congruence on $\mathcal{B}$ such that all, Boolean extensions of, the equivalence classes are isomorphic with $\mathcal{I}_{\tau}$. Let f and g be those isomorphisms from $[\psi]$ and $[\varphi]$ onto $\mathcal{I}_{\tau}$, respectively. Intuitively, the next proposition states that if $[\psi] \leq [\varphi]$, then there is a $\varphi' \in [\varphi]$ with $f(\psi) = g(\varphi')$ such that $\psi \leq^{\Delta} \varphi'$. The expression, $\psi \equiv \varphi' \pmod{\sim_{\tau}}$ designates the same relation between ψ and φ'.

PROPOSITION 6.12: $[\psi] \leq [\varphi] \Leftrightarrow \exists \varphi' \in [\varphi]: \psi \leq^{\Delta} \varphi'$

Proof: $\Rightarrow$: I claim my proposition obtains for $\varphi' := (\varphi \wedge \neg\tau) \vee (\psi \wedge \tau)$. Consequently, I must prove that 1. $\varphi' \in [\varphi]$ that is $\varphi' \leftrightarrow \neg\varphi \vDash \tau$; and 2. $[\psi] \leq [\varphi]$ implies $\psi \leq^{\Delta} \varphi'$. 1. $\varphi' \leftrightarrow \neg\varphi \vDash \tau$; $\Leftrightarrow$ $(\varphi' \wedge \neg\varphi) \vee (\neg\varphi' \wedge \varphi) \vDash \tau$; $\Leftrightarrow$ $(\psi \wedge \tau \wedge \neg\varphi) \vee (\neg(\varphi \wedge \neg\tau) \wedge \neg(\psi \wedge \tau) \wedge \varphi) \vDash \tau$; $\Leftrightarrow$ $(\psi \wedge \tau \wedge \neg\varphi) \vee (\varphi \wedge \tau \wedge (\neg\psi \vee \neg\tau)) \vDash \tau$; $\Leftrightarrow$ $(\psi \wedge \tau \wedge \neg\varphi) \vee (\varphi \wedge \tau \wedge \neg\psi) \vDash \tau$, which clearly holds. Ad 2. I must prove that $\psi \vDash \varphi \vee \tau$ implies a. $\psi \vDash \varphi' \vee \tau$ and b. $\varphi' \wedge \tau \vDash \psi$. a. If $\psi \vDash \varphi \vee \tau$ then $\psi \vDash (\varphi \vee \tau) \vee (\psi \wedge \tau) \Leftrightarrow \psi \vDash (\varphi \wedge \neg\tau) \vee (\psi \wedge \tau) \vee \tau \Leftrightarrow \psi \vDash \varphi' \vee \tau$. b. $\varphi' \wedge \tau \vDash \psi$ clearly holds as $((\varphi \wedge \neg\tau) \vee (\psi \wedge \tau)) \wedge \tau \vDash \psi$ equals $(\psi \wedge \tau) \vDash \psi$. $\Leftarrow$: $\psi \leq^{\Delta} \varphi'$ implies $[\psi] \leq [\varphi']$ and $\varphi' \in [\varphi]$, thus $[\psi] \leq [\varphi]$. ⊠

6.3.4. Language Dynamic Behaviour of Two Content Definitions

Finally, I consider the differences between the language dynamic behaviour of the Δ-definition and the Cn-proposal. To begin with, recall that $\psi \sim^{+} \varphi$ is the same as $\psi \sim_{\tau} \varphi$ (obs. 1.5, p. 20). Furthermore, I observe that $\psi \sim^{+} \varphi$ and $\psi \leq^{\Delta} \varphi$ are unrelated.

Observation 6.13: a. Suppose the truth τ is complete; then $\psi \sim^{+} \varphi$ implies $\psi \leq^{\Delta} \varphi$ or $\varphi \leq^{\Delta} \psi$; b. if τ is incomplete $\sim^{+}$ is indifferent to $\leq^{\Delta}$ i.e. if $\psi \sim^{+} \varphi$ then $\psi \leq^{\Delta} \varphi$ or $\varphi \leq^{\Delta} \psi$ or $\psi \nleq\ngeq^{\Delta} \varphi$.

Proof: a. Let τ be complete. Then, if $\psi \sim^{+} \varphi$ either 1. $\tau \vDash \psi$, $\varphi \vDash \neg\tau$, and $\psi \equiv \varphi \vee \tau$, and therefore $\psi \leq^{\Delta} \varphi$, or 2. $\tau \vDash \varphi$, $\psi \vDash \neg\tau$, and $\varphi \equiv \psi \vee \tau$, and therefore $\varphi \leq^{\Delta} \psi$. b. If $\psi \sim^{+} \varphi$ and τ is not complete, 1. and 2. may obtain; however, they also may be uncomparable. Assume that $\psi \leftrightarrow \neg\varphi \equiv \tau$ (*), which implies $\psi \sim^{+} \varphi$, and assume $\varphi \nvDash \psi$ (†), and $\psi \nvDash \varphi$ (‡). As the first assumption (*) is equivalent to $\mathrm{Mod}(\psi)\ \Delta\ \mathrm{Mod}(\varphi) = \mathrm{Mod}(\tau)$, it implies $\mathrm{Mod}(\psi) \subseteq \mathrm{Mod}(\varphi) \cup \mathrm{Mod}(\tau)$, that is

$\psi \vDash \varphi \vee \tau$. In the same way (*) implies $\varphi \vDash \psi \vee \tau$. (†) implies there is a $\mathfrak{M} \in \text{Mod}(\varphi)$ such that $\mathfrak{M} \notin \text{Mod}(\psi)$; since (*) implies that $\mathfrak{M} \in \text{Mod}(\tau)$ $\varphi \wedge \tau \nvDash \psi$. In the same way it follows form (‡) and (*) that $\psi \wedge \tau \nvDash \varphi$. Thus, $\psi \sim^{+} \varphi$ obtains together with $\psi \not\asymp^{\Delta} \varphi$. ⊠

An example of b. of the preceding observation reads: $\tau := p$, $\psi := p \leftrightarrow q$, and $\varphi := \neg q$. Further, note that $\psi \leq^{\Delta} \varphi$ does not imply $\varphi \nleq^{+} \psi$. Let us assume that in $\mathscr{L}[p,q]$ the truth is $p \wedge q$. Then, for instance, $p \vee q \leq^{\Delta} p \leftrightarrow \neg q$ and hence $p \vee q \leq^{+} p \leftrightarrow \neg q$. Yet, since $p \leftrightarrow \neg q \sim^{+} p \vee q$, the +-definition also implies $p \leftrightarrow \neg q \leq^{+} p \vee q$.

The preceding observation illustrates the fact that the Δ-definition and the Cn-proposal behave differently under logical strengthening of the truth. According to the Cn-proposal logical strength of the truth covaries with the strength of the ordering. Logical weakening of the truth decreases $\mathscr{B}/I_{\tau}$, and the growth of the equivalence classes increases the number of comparable sentences. The behaviour of the Δ-definition neither varies nor covaries with the strength of the truth, and displays another behaviour than the +-definition. For instance, the $\leq^{\Delta}$-order of $\mathscr{B}$ for $\mathscr{L}[p,q]$ with $\tau := p \wedge q$ is isomorphic to the $\leq^{\Delta}$-order with $\tau := p \vee q$. Finally, let us consider context dependency of the Cn-proposal.

DEFINITION 6.6: A *preorder* $\leq$ on Prop($\mathscr{L}$) is *weakly (strongly) context independent* iff for all $\psi, \varphi \in \text{Prop}(\mathscr{L})$ and all $\mathscr{L}' \supset \mathscr{L}$, with ψ', φ' being translations of ψ, φ: If $\psi \leq \varphi$, and $\varphi \nleq \psi$, then $\varphi' \nleq \psi'$ ($\psi' \leq \varphi'$).

Observation 6.14: The $\leq^{+}$-(pre)order is *weakly context independent.*

Proof: Suppose $\psi \leq^{+}_{\tau} \varphi$ and $\varphi \nleq^{+}_{\tau} \psi$ (†); I now have to prove that $\varphi' \nleq^{+}_{\tau} \psi'$. By definition, (†) means $\varphi \nvDash \psi \vee \tau_{\mathscr{L}}$. Hence, there is a $\mathfrak{M} \in \text{Mod}(\varphi)$ with $\mathfrak{M} \notin \text{Mod}(\psi \vee \tau_{\mathscr{L}})$. Let $\mathfrak{M}'$ be an $\mathscr{L}'$-expansion of $\mathfrak{M}$—that is $\mathfrak{M}' \restriction_{\mathscr{L}} = \mathfrak{M}$—and ψ' and φ' be the translations of ψ and φ. Then, the adequacy of the translation implies $\mathfrak{M}' \vDash \varphi'$; and as $\tau_{\mathscr{L}'} \vDash \tau'_{\mathscr{L}}$ and $\mathfrak{M}' \nvDash \psi' \vee \tau'_{\mathscr{L}}$ together imply $\mathfrak{M}' \nvDash \psi' \vee \tau_{\mathscr{L}'}$, proving $\varphi' \nleq^{+}_{\tau} \psi'$. ⊠

Observation 6.15: The $\leq^{+}$-(pre)order is not strongly context independent.

Proof: Consider the following counter example; in $\mathscr{L}[p,q]$, $p \leq^{+}_{p \wedge q} p \leftrightarrow \neg q$ but in $\mathscr{L}[p,q,r]$ $p \nleq^{+}_{p \wedge q \wedge r} p \leftrightarrow \neg q$, since $p \wedge q \wedge \neg r \vDash p$ but $p \wedge q \wedge \neg r \nvDash (p \leftrightarrow \neg q) \vee (p \wedge q \wedge r)$. In this example $\psi \leq^{+}_{\tau} \varphi$ and $\varphi' \nleq^{+}_{\tau'} \psi'$ obtain and $\psi' \leq^{+}_{\tau'} \varphi'$ fails to hold. ⊠

Interestingly, a relevant extension of the language in which the logical strength increases may affect the verisimilitude relation between theories. An irrelevant extension for which $\tau_{\mathscr{L}'} = \tau'_{\mathscr{L}}$ does not change the original verisimilitude relation. Perhaps this is an indication of the relevance of verisimilitude for language dynamic purposes.

In subsection 6.3.1 we encountered the notions of an ideal I_τ, the congruence $\sim_\tau$, and the quotient algebra $\mathscr{B}/I$; and in subsection 6.3.2 we saw that the $\leq^+$-order of a Lindenbaum algebra $\mathscr{B}$ is isomorphic to the quotient algebra $\mathscr{B}/I_\tau$ (prop. 6.11, p. 207). Consequently, the $\leq^+$-order is a preorder of $\mathscr{B}$, partially ordering the equivalence classes of the $\sim_\tau$-congruence established by I_τ. This is the reason why the $\leq^+$-order, in contrast to the $\leq^\Delta$-order, is not truth-value dependent. Next, the +-definition is a radical content proposal. Irrespective of the truth-values of φ and ψ, if $\varphi \vDash \psi$ and $\psi \nvDash \varphi$, then ψ cannot be more verisimilar than φ. Consequently, the +-definition suffers from the child's-play objection, and, similar to the Δ-proposal, Miller's extensional substitution argument does not affect the $\leq^+$-order. As to the strength of the ordering relations, the removal of the An-clause of the Δ-definition means for all $\psi, \varphi \in \mathscr{B}$, $\psi \leq^\Delta \varphi$ implies $\psi \leq^+ \varphi$. In subsection 6.3.3 we saw that for those ψ with $\psi \equiv \varphi \pmod{\sim_\tau}$, the reverse implication also obtains (prop. 6.12. p. 210). Further, like the Δ-definition the +-definition applies for complete and incomplete truths. Its strength of comparison covaries with the strength of the truth, and the strength of the Δ-comparison neither varies nor covaries with the strength of the truth. Finally, under a relevant extension of the language, the $\leq^+$-order is weakly but not strongly context independent.

Towards the end of the preceding section (p. 201), I anticipated the explanation of the congruence between the two layers at the left hand side of Figure 3. The present section provided the explanation. I have "put together the layers of true and false propositions" by replacing the Δ-proposal by the $\leq^+$-order, which partially orders equivalence classes of $\mathscr{B}$. In the next section, we will come to my second announcement.

6.4. REFINED VERISIMILITUDE: $\leq^+_{\precsim}$

In this section, I present my most important contribution to the approach-to-the-truth research. My refined verisimilitude definition partially orders the equivalence classes that result from merging the partition of the $\leq^+$-order with that of my truthlikeness order. In addition to the partition brought about by the truth ideal, a second partition puts two propositions in the same equivalence class if, roughly speaking, their constituents have the same height in the constituent algebra. This establishes a truthlikeness order $\precsim_h$, which is stronger than the $\preceq_h$-order of subsection 6.2.2. This $\precsim_h$-order can be combined with the $\leq^+$-order as the second is order-isomorphic with the quotient algebra $\mathscr{B}/I$. As a preliminary, I first specify what it is for a partial order to be compatible with an arbitrary equivalence relation. Then I define the truthlikeness preorder based on the assumption that all atomic propositions are equally important. According to this preorder, a uniform substitution of the atomic propositions does not change the truthlikeness of a proposition. Finally, I introduce my combination of the $\leq^+$-order and my truthlikeness preorder, and finish with the examination of some the properties of this new refined verisimilitude ordering.

6.4.1. *Compatibility Ordering and Partition*

Assume $\mathscr{A} := \langle A; \leq \rangle$ is a partially ordered set, and $\sim$ is an equivalence relation on A. In what follows, we need an answer to the question under what conditions the partial order $\leq$ and the equivalence relation $\sim$ are compatible and may be merged into a partial order on $A/\sim$, or, which is the same, a preorder on A. To answer this question, I may first consider a clear-cut situation in which a partial order on A induces a partial order on the equivalence classes of A. Let us call $\leq^{\sim}$, defined by

(2) $\quad [a] \leq^{\sim} [b] \;:=_{def}\; a \leq b$ and $a \nsim b$,

the *representative of* $\leq$ *on* $A/\sim$. (2) uniquely determines $\leq^{\sim}$ iff

(3) For all four-tuples $\langle a, b, x, y \rangle \in A^4$ with $a \sim x$ and $b \sim y$, $a \leq b$ implies $x \leq y$

Under these circumstances definition (2) is *independent of the representatives a and b*. Of course, my compatibility definition must include this clear-cut situation. Additionally, it must include situations that may be extended into the preceding situation.

DEFINITION 6.7: The partial $\leq$-order on A *is compatible with an equivalence* $\sim$ on A if for all $\langle a, b, x, y \rangle \in A^4$ such that $a \sim x$, $b \sim y$, $a \nsim b$ holds: if $a \leq b$, then $y \nleq x$.
Notation: $\mathrm{Comp}_A(\sim, \leq)$

Definition 6.7 relaxes constraint (3) in two respects. It admits situations in which $a \sim x$, $b \sim y$, and $a \sim b$, that $a \leq b$, and $y \leq x$; and it allows that $a \sim x$, $b \sim y$, $a \leq b$, and $x \nleq y$. Constraint (3) excludes both situations. The next observation illustrates definition 6.7. It proves that, as was to be expected, the truth congruence $\sim_\tau$ of the preceding subsection is compatible with the $\leq^{\Delta}$-order of a Lindenbaum algebra $\mathscr{B}$;

Observation 6.16: Let $\mathscr{B}$ be a Lindenbaum algebra; then $\mathrm{Comp}_{\mathscr{B}}(\sim^{+}, \leq^{\Delta})$
Proof: Let us assume that $\psi \sim^{+} \psi'$, $\varphi \sim^{+} \varphi'$, $\psi \nsim^{+} \varphi$, and $\psi \leq^{\Delta} \varphi$; the last assumption implies $[\psi] \leq [\varphi]$ in $\mathscr{B}/\mathrm{I}_\tau$. Furthermore, assume that also $[\varphi'] \leq [\psi']$; since $\leq$ is a partial ordering on $\mathscr{B}/\mathrm{I}_\tau$ this would imply that $[\psi] = [\varphi]$, which contradicts $\psi \nsim^{+} \varphi$. Hence, we must conclude $[\varphi'] \nleq [\psi']$. ⊠

PROPOSITION 6.17: Let $\leq$ partially order A, and $\sim$ be an equivalence relation on A; then $\mathrm{Comp}_A(\sim, \leq)$ implies that all $[b]\sim \subset A$ are convex regarding $\rho := \overline{\leq \textit{ or } \sim}$.

Proof: We have to prove that every equivalence class $[b]\sim$ that fails to be convex falsifies $\mathrm{Comp}_A(\sim, \leq)$. Suppose some $[b]\sim$ fails to be convex regarding ρ; This means $\exists a, \exists x \in [b]\sim$, $\exists y \in A: y \notin [b]\sim$ with a $\rho\, y\, \rho\, x$ or $x\, \rho\, y\, \rho\, a$. Although $a \sim x$, $y \sim y$ and $a \not\sim x$, from $x \not\sim y \not\sim a$, it follows that $a \leq y \leq x$ or $x \leq y \leq a$, both falsifying $\mathrm{Comp}_A(\sim, \leq)$. ⊠

The last observation of this subsection demonstrates that if $\sim$ and $\leq$ are compatible on A, then their transitive closure uniquely determines $\leq\sim$ on $A/\sim$.

Observation 6.18: Let $\leq$ be a partial order and $\sim$ be an equivalence on A and let $\mathrm{Comp}_A(\sim, \leq)$. If $\lesssim$ denotes the transitive closure $\overline{\leq \text{ or } \sim}$ then $[a]\sim \leq^{\sim} [b]\sim$ $:=_{def} a \lesssim b$ and $a \not\sim b$, uniquely determine $\leq\sim$ on $A/\sim$.

Proof: The implication immediately follows from (2) and (3), p. 213. Let $a \sim x$ and $b \sim y$; then $a \lesssim b$ implies $x \lesssim a \lesssim b \lesssim y$; hence $x \lesssim y$.

6.4.2. Strengthening the Δ-Likeness

A comparative truthlikeness refinement $\preceq_h$ was introduced in subsection 6.2.2 according to which, for instance, $p \wedge q \wedge \neg r$ is more like the truth $p \wedge q \wedge r$, than $\neg p \wedge \neg q \wedge \neg r$. This order left $\neg p \wedge r$ and $p \wedge \neg q$, or even, $\neg p \wedge q$ and $p \wedge \neg q$ uncomparable. Figure 2 (p. 198) illustrates the result of this weakness for $\mathscr{L}[p,q]$. Now we say that two propositions have the same disjunctive normal form if a series of uniform substitutions of free atomic propositions can make them equivalent.[17] The diagram of Figure 2 illustrates that many propositions with the same logical form remain uncomparable. In this subsection the strength of the truthlikeness refinement is increased such that propositions of the same logical form become equally truthlike. Relating truthlikeness to logical form rather than atomic propositions considerably increases the strength of the definition. For the sake of simplicity in this subsection it is assumed that all atomic propositions are equally important. Concrete examples may involve an order of importance of the atomic propositions.

I start with the definition of the truthlikeness preorder on $\mathscr{B}$, assuming that the truth is complete and can be represented by the conjunction of all atomic propositions, this assumption also holds for the sequel of this section.

DEFINITION 6.8: Let ψ and φ be propositions of $\mathscr{L}$. Then ψ *is at least as similar to the truth* as φ iff $\exists\mu$: $\mathrm{Cnst}(\psi) \rightarrowtail \mathrm{Cnst}(\varphi)$ satisfying the following conditions:
1. μ is a bijection
2. For all $\alpha \in \mathrm{Cnst}(\psi)$: $|\mathrm{pl}(\alpha)| \geq |\mathrm{pl}(\mu(\alpha))|$

Notation: $\psi \precsim_h \varphi$ (in which $h = |\mu|$);[18] ($\psi \prec_h \varphi$ iff $|\mathrm{pl}(\alpha)| > |\mathrm{pl}(\mu(\alpha))|$)

The comparative truthlikeness order of definition 6.3 (p.197), $\psi \preceq_h \varphi$, implies $\psi \precsim_h \varphi$ since $\mathrm{pl}(\mu(\alpha)) \subseteq \mathrm{pl}(\alpha)$ straightforwardly implies $|\mathrm{pl}(\alpha)| \geq |\mathrm{pl}(\mu(\alpha))|$.

Again, h indicates the logical strength of the propositions. Generally, $\precsim_h$ on $\mathscr{B}$ is a preorder. It is *reflexive and transitive* (proof is left to the reader), but it is not anti-symmetric. For example, $p \wedge \neg q \precsim_h \neg p \wedge q$, and $\neg p \wedge q \precsim_h p \wedge \neg q$ obtain both. More particularly, in addition to forming a preorder, all constituents are also connected. Suppes (1957, p. 222) calls $\precsim_l$ a *weak ordering.*[19] We abbreviate "$\psi \precsim_h \varphi$ and $\varphi \precsim_h \psi$" by $\psi \simeq \varphi$. The $\simeq$-relation is an equivalence relation and partitions $\mathscr{B}$. As $\precsim_h$ preorders $\mathscr{B}$, $\precsim_h^{\simeq}$ defined by $[\psi]^{\simeq} \precsim_h^{\simeq} [\varphi]^{\simeq} :=_{def} \psi \precsim_h \varphi$, partially orders $\mathscr{B}/\simeq$; and since $\precsim_l$ is connected in the set of constituents $\precsim_l^{\simeq}$ linearly orders the $\simeq$-equivalence classes of the constituents.

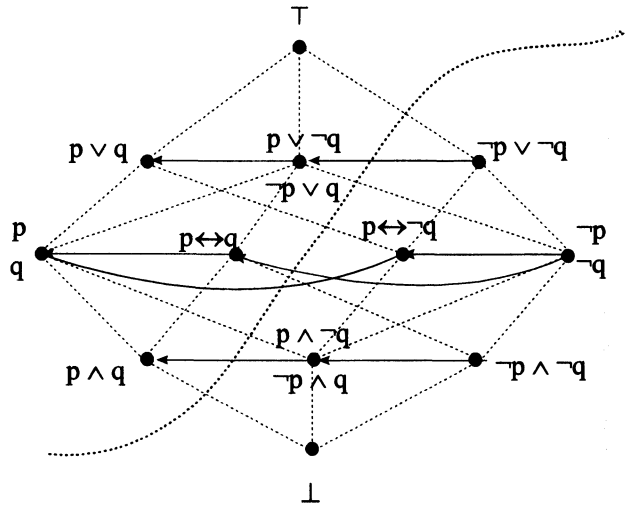


Fig. 8. $\mathscr{B}[p,q]/\simeq_h$

Let us consider the situation in the two propositional example. Figure 7 depicts the quotient algebra $\mathscr{B}/\simeq$. The nodes represent the $\simeq$-equivalence classes of $\mathscr{B}/\simeq$, in the two propositional case, these classes are singletons and pairs; the dotted lines stand for the covering relation (p. 196) in $\mathscr{B}/\simeq$; and the continuous horizontal lines and "bows" depict the $\precsim_l^{\simeq}$-relation in $\mathscr{B}/\simeq$. If two nodes are connected the elements on the left side are closer to $p \wedge q$ than those on the right side. The diagram illustrates that, whereas $\precsim_h$ linearly orders the equivalence classes of the constituents, and their negations, it also partially orders the equivalence classes of the other propositions of $\mathscr{B}$.

PROPOSITION 6.19: The $\precsim_h$-definition is *strongly context independent.*

Proof: We must prove that if $\psi \precsim_h \varphi$ in $\mathscr{L}$, then $\psi' \precsim_h \varphi'$ obtains for all $\mathscr{L}' \supseteq \mathscr{L}$. Suppose $\psi \precsim_h \varphi$ for a finite propositional language $\mathscr{L}$; then, there is a bijection μ:

Cnst(ψ) $\rightarrowtail$ Cnst(φ) such that for all $\alpha \in$ Cnst(ψ): $|\mathrm{pl}(\alpha)| \geq |\mathrm{pl}(\mu(\alpha))|$ (*). Let $\mathscr{L}' \supseteq \mathscr{L}$ be a finite propositional language, such that for language $\mathscr{L}''$ voc($\mathscr{L}''$) = voc($\mathscr{L}'$)-voc($\mathscr{L}$) holds. Then, Cnst($\top''$) consist of 2^n constituents β_j if $|$voc($\mathscr{L}''$)$|$ = n. Obviously, for all $\alpha \in$ Cnst($\top$) and $\beta \in$ Cnst($\top''$), $\alpha \wedge \beta$ is a constituent of $\mathscr{L}'$, and $\psi' = \vee_i \vee_j(\alpha_i \wedge \beta_j)$ is the translation of all $\psi = \vee_i\alpha_i \in$ Prop($\mathscr{L}$), if β_j ranges over all Cnst($\top''$). Hence (*) implies for all $\alpha_i \wedge \beta_j \in$ Cnst(ψ') that $|\mathrm{pl}(\alpha_i \wedge \beta_j)| \geq |\mathrm{pl}(\mu'(\alpha_i \wedge \beta_j))|$ if $\mu'(\alpha_i \wedge \beta_j) := \mu(\alpha_i) \wedge \beta_j$; and $\psi' \precsim_h \varphi'$. ⊠

As a preliminary to the next subsection, I consider the representative of $\precsim_h$ on $\mathscr{B}/I_\tau$. Since the Boolean extension of Cn(τ) is isomorphic with $\mathscr{B}/I_\tau$, I define this representative by ($[\psi]\sim$ being elements of $\mathscr{B}/I_\tau$)

(4) $[\psi]\sim \precsim [\varphi]\sim \; :=_{def} \; \psi \vee \tau \precsim_h \varphi \vee \tau$

As $\psi \precsim_h \varphi$ implies $h(\psi) = h(\varphi)$, the height of $[\psi]$ and $[\varphi]$ in $\mathscr{B}/I$ are the same if $[\psi]\sim \precsim [\varphi]\sim$. Definition (4) is trivially independent of φ and ψ, since $\psi \vee \tau \in [\psi]\sim$ is unique. Further, the $\precsim$-relation not only establishes a preorder on $\mathscr{B}$, it also preorders $\mathscr{B}/I_\tau$. Again, the order is *reflexive and transitive* but not antisymmetric. For instance, for the two different equivalence classes $[p \wedge \neg q]\sim \precsim [\neg p \wedge q]\sim$ and $[\neg p \wedge q]\sim \precsim [p \wedge \neg q]\sim$ obtain both although $\neg p \wedge q \notin [p \wedge \neg q]\sim$. $[\psi]\sim \simeq [\varphi]\sim$ stands for "$[\psi]\sim \precsim [\varphi]\sim$ and $[\varphi]\sim \precsim [\psi]\sim$". This $\simeq$-relation is symmetric by definition, and inherits reflexivity and transitivity from $\precsim$; it is an equivalence relation on $\mathscr{B}/I_\tau$. Thus, in the next paragraph I consider $(\mathscr{B}/I_\tau)/\simeq$.

DEFINITION 6.9: Let $\mathscr{B}/I_\tau$ be the quotient algebra of a Lindenbaum algebra $\mathscr{B}$; then I will abbreviate $(\mathscr{B}/I_\tau)/\simeq$ by $\mathscr{B}/\approx$.

We have encountered two partitions of $\mathscr{B}$. The congruence of the truth-ideal was the first; it yielded the $[\varphi]\sim$ equivalence classes. The preorder $\precsim_h$ established the second. It produces the $[\varphi]^{\simeq}$ equivalence classes. It is an elementary exercise to combine the two partitions into one coarser partition. The $[]\sim$-partition is finer than the $[\![]\!]^{\approx}$-partition. That is, every $[\![]\!]^{\approx}$-set contains at least one $[]\sim$-set and no $[]\sim$-set belongs to more than one $[\![]\!]^{\approx}$-set.[20] Thus, $[\psi]\sim = [\varphi]\sim$ implies, and is not implied by $[\![\psi]\!]^{\approx} = [\![\varphi]\!]^{\approx}$. If $\sim$ and $\simeq$ are the truth-ideal and the truthlikeness equivalences, then $\psi \approx \varphi$ means that $\psi \; \overline{\sim \text{ or } \simeq} \; \varphi$ and $\psi \in [\![\varphi]\!]^{\approx}$. In the next subsection I shall present the most natural way to order the elements of $\mathscr{B}/\approx$.

6.4.3. Refined Verisimilitude

In the preceding subsections I prepared the grounds for my *refined verisimilitude* definition. The consequence definition resulted in the order of the quotient algebra $\mathscr{B}/I_\tau$. Next, the likeness refinement mentioned in the preceding subsection ordered the equivalence classes of $\mathcal{B}/I_\tau$ of equal heights. This resulted in a coarser partitioning of $\mathcal{B}$, $\mathcal{B}/\approx$. Schematically, the procedure may be summarized as follows:

$$\mathcal{B} \oplus \leq^+ \;\Rightarrow\; \mathcal{B}/I_\tau \oplus \precsim \;\Rightarrow\; \mathcal{B}/\approx$$

In this subsection we will see that the transitive closure of the representatives of $\leq^+$ and $\precsim$ on $\mathcal{B}/\approx$ naturally orders the $[\![]\!]$-elements of $\mathcal{B}/\approx$. This closure intuitively mirrors the growth in depth, $\leq^+$, and breadth, $\precsim$, mentioned in Chapter 1 (p. 2). The definition procedure consists of four steps. The first step is to transfer the $\leq^+$-order of $\mathcal{B}/I_\tau$ to $\mathcal{B}/\approx$; next, the $\precsim_h$-relation must be transferred to $\mathcal{B}/\approx$; then I have to show that these representatives $\precsim^\approx$ and $\leq^\approx$ of $\precsim$ and $\leq_{\mathcal{B}/I}$ on $\mathcal{B}/\approx$ are compatible; and finally $\precsim^\approx$ and $\leq^\approx$ are merged into $\leq^+_\precsim$. The subsection ends with an examination of some properties of my refined verisimilitude proposal. As a preliminary to these four steps, it is shown that the equivalence relation introduced by the $\precsim$ order is compatible with the $\leq^+$-order on $\mathcal{B}/I$.

PROPOSITION 6.20: The $\simeq$-relation and $\leq^+$-order are compatible on $\mathcal{B}$

Proof: Let $[\psi] \simeq [\xi]$, $[\varphi] \simeq [\chi]$, $[\psi] \neq [\varphi]$, and let $[\psi] \leq [\varphi]$. The last two assumptions imply that $\psi \vee \tau \vDash \varphi \vee \tau$ and $\varphi \vee \tau \nvDash \psi \vee \tau$; thus $h(\psi \vee \tau) < h(\varphi \vee \tau)$. Additionally, $[\psi] \simeq [\xi]$ and $[\varphi] \simeq [\chi]$ imply that $h(\psi \vee \tau) = h(\xi \vee \tau)$ and $h(\chi \vee \tau) = h(\varphi \vee \tau)$. Consequently, $h(\xi \vee \tau) < h(\chi \vee \tau)$, and thus $[\chi] \not\leq_{\mathcal{B}/I} [\xi]$. ⊠

The preceding proof shows that all partitions putting only elements of $\mathcal{B}/I$ of the same height into one equivalence class are compatible with the order of the quotient algebra $\mathcal{B}/I$. Hence, we may take the first step towards our final comparative approach-to-the-truth definition, and combine the $\simeq$ and $\leq_{\mathcal{B}/I}$ relations on $\mathcal{B}/I$.

DEFINITION 6.10: Let $\psi, \varphi \in Prop(\mathcal{L})$; then $[\![\psi]\!] \leq^\approx [\![\varphi]\!]$ iff there is a $[\psi']$ in $[\![\psi]\!]$, and a $[\varphi']$ in $[\![\varphi]\!]$ such that $[\psi'] \leq_{\mathcal{B}/I} [\varphi']$.

Obviously, the $\leq^\approx$-definition is independent of the representatives. Consider the set $\mathcal{B}/\approx$ ordered by $\leq^\approx$ and the function $g: \mathcal{B}/I \to \mathcal{B}/\approx$ defined by $g([\psi]) = [\![\psi]\!]$. Let us define the $\wedge^\approx$ and the $\vee^\approx$ in $\mathcal{B}/\approx$.

$$[\![\varphi]\!] \wedge^\approx [\![\psi]\!] = [\![\psi]\!] :=_{def} [\![\psi]\!] \leq^\approx [\![\varphi]\!]$$
$$[\![\varphi]\!] \vee^\approx [\![\psi]\!] = [\![\varphi]\!] :=_{def} [\![\psi]\!] \leq^\approx [\![\varphi]\!]$$

First we may observe that g is not an homomorphism. For instance, in the case of two propositions, $g([p] \vee_{\mathcal{B}/I} [q]) = g([p \vee q]) = [\![p \vee q]\!] \neq [\![p]\!] \vee^\approx [\![q]\!] = g([p]) \vee^\approx g([q])$ (cf. Figure 8). Next we observe that although $\mathcal{B}/\approx$ is not a Boolean Algebra it still is a *lattice* since for all φ and ψ, $[\![\varphi]\!] \wedge^\approx [\![\psi]\!]$ and $[\![\varphi]\!] \vee^\approx [\![\psi]\!]$ are the *glb* and the *lub* in $\mathcal{B}/\approx$, respectively. This contention is easily verified. If $[\![\varphi]\!] \neq [\![\psi]\!]$ and (therefore) $[\varphi] \neq [\psi]$, and in case $[\![\varphi]\!] = [\![\psi]\!]$ and $[\varphi] = [\psi]$, mapping g is *lub* and *glb* preserving; and in case $[\![\varphi]\!] = [\![\psi]\!]$ and $[\varphi] \neq [\psi]$, $[\![\varphi]\!] \wedge^\approx [\![\psi]\!] = [\![\psi]\!] = [\![\varphi]\!] \vee^\approx [\![\psi]\!]$. Consequently, all $[\![\varphi]\!]$ and $[\![\psi]\!]$ have a *lub* and a *glb* in $\mathcal{B}/\approx$.

Figure 8 illustrates my first step the refined verisimilitude definition and shows the compatibility of the $\approx$-equivalence classes (the ellipses) and the $\leq^{+}$-order on $\mathcal{B}/\approx$ in case of the two atomic propositions and the truth $p \wedge q$. Now that I have defined a representative of $\leq^{+}$ on $\mathcal{B}/\approx$, the second step towards the refined verisimilitude definition is to transfer the $\precsim$-relation to $\mathcal{B}/\approx$.

DEFINITION 6.11: Let ψ, $\varphi \in \mathrm{Prop}(\mathcal{L})$; then $[\psi] \precsim^{\approx} [\varphi]$ if and only if $[\psi] \precsim [\varphi]$

The $\precsim^{\approx}$-relation *partially orders* $\mathcal{B}/\approx$ because the reflexivity and transitivity of $\precsim_h$ guarantees $\precsim^{\approx}$ to be reflexive and transitive, and $[\psi] \precsim^{\approx} [\varphi]$ and $[\varphi] \precsim^{\approx} [\psi]$ imply $[\psi] \approx [\varphi]$.

PROPOSITION 6.21: The preceding definition is independent of $[\psi]$ and $[\varphi]$.

Proof: Let $[\psi] \approx [\xi]$, $[\varphi] \approx [\chi]$, and let $[\psi] \precsim [\varphi]$. The last assumption implies $\psi \vee \tau \precsim \varphi \vee \tau$. The first two assumptions imply $\psi \vee \tau \approx \xi \vee \tau$ and $\varphi \vee \tau \approx \chi \vee \tau$; thus $\xi \vee \tau \precsim \chi \vee \tau$ and therefore $[\xi] \precsim [\chi]$. ⊠

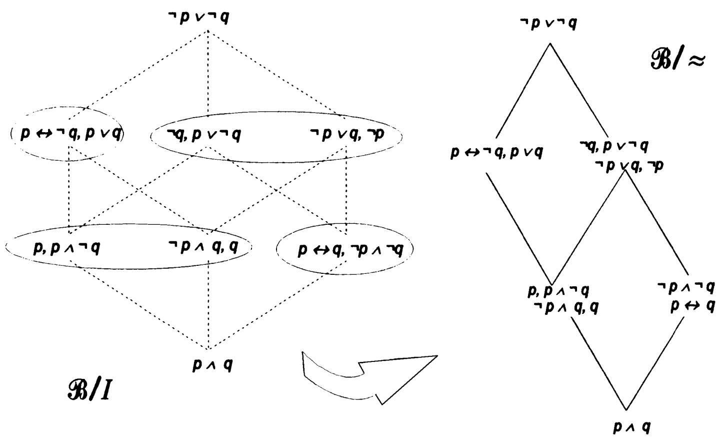


Fig. 9. The $\leq^{\approx}$-relation on $\mathcal{B}/\approx$

The third and last preparatory step towards the definition is to show that $\precsim^{\approx}$ and $\leq^{\approx}$, the representatives of $\precsim$ and $\leq_{\mathcal{B}/I}$ on $\mathcal{B}/\approx$, are compatible. Recall that partial order $\preceq$ is compatible with $\leq$ iff their irreflexive companions are compatible.

Lemma 22: $\precsim^{\approx}$ and $\leq^{\approx}$ are compatible on $\mathcal{B}/\approx$
Proof: I must show that for all ψ, $\varphi \in \mathrm{Prop}(\mathcal{L})$ $[\psi] \precsim^{\approx} [\varphi]$ implies $\neg([\varphi] <^{\approx} [\psi])$. Suppose $[\psi] \precsim^{\approx} [\varphi]$; then $[\psi] \precsim [\varphi]$ and $h_{\mathcal{B}/I}([\psi]) = h_{\mathcal{B}/I}([\varphi])$ (p. 216). Therefore, for all $[\varphi]$, and $[\varphi']$, if $[\varphi] \approx [\varphi']$, then $h_{\mathcal{B}/I}([\varphi]) = h_{\mathcal{B}/I}([\varphi'])$; thus $h_{\mathcal{B}/I}([\varphi]) = h_{\mathcal{B}\approx}([\varphi])$, and $h_{\mathcal{B}\approx}([\psi]) = h_{\mathcal{B}\approx}([\varphi])$. Consquently $\neg([\varphi] <^{\approx} [\psi])$. ⊠

Since we know that the representatives of $\precsim$ and $\leq^+$ on $\mathscr{B}/\approx$ are compatible, I can take the fourth step and combine $\precsim^{\approx}$ and $\leq^{\approx}$ into my refined verisimilitude definition:

DEFINITION 6.12: ψ is *at least as (refined) verisimilar* as φ iff

$[\![\psi]\!]\ \overline{\leq^{\approx} \text{or} \precsim^{\approx}}\ [\![\varphi]\!]$

Notation: $\psi \leq^+_{\precsim} \varphi$

Observation 6.22: Let φ and ψ be propositions of $\mathscr{L}$; then $\psi \leq^+_{\precsim} \varphi \Leftrightarrow \psi\ \overline{\leq^+ \text{or} \precsim_h} \varphi$

Proof: $\Rightarrow$: As all propositions in $[\![\psi]\!]$ are connected by $\simeq$ or $\sim$-chains, $\psi \leq^+_{\precsim} \varphi$ implies there is a $\leq^+$ or $\precsim_h$-chain from ψ to φ. $\Leftarrow$: First, let us assume that 1. $\psi \leq^+ \varphi$ or 2. $\psi \precsim_h \varphi$. 1. means a. $\psi \equiv \varphi \vee \tau$ or b. $\psi \vDash_{\neq} \varphi \vee \tau$. If 1.a then $[\psi] = [\varphi]$ and $\psi \leq^+_{\precsim} \varphi$; if 1.b then $[\psi] <_{\mathscr{B}/I} [\varphi]$ implying $[\![\psi]\!] \leq^{\approx} [\![\varphi]\!]$, and $\psi \leq^+_{\precsim} \varphi$. Second, 2. implies that a. $\psi \approx_h \varphi$ or b. $\psi \precsim_h \varphi$ and $\neg(\varphi \precsim_h \psi)$. If 2.a then $[\![\psi]\!] = [\![\varphi]\!]$; if 2.b then $[\![\psi]\!] \precsim^{\approx} [\![\varphi]\!]$ and both imply $\psi \leq^+_{\precsim} \varphi$. Hence the implication holds if the length of the chain of the transitive closure is one; repetition of the preceding argument proves the statement for any finite chain length of the transitive closure. ⊠

Let me summarize the preceding steps. I showed that the representatives of $\leq^+$ and $\precsim_h$ on $\mathscr{B}/\approx$ are compatible, and partially order $\mathscr{B}/\approx$. I merged the two into the refined verisimilitude order of Prop($\mathscr{L}$), which is equivalent to the transitive closure of the "$\leq^+$ or $\precsim_h$" disjunction on Prop($\mathscr{L}$). Figure 9 illustrates how the $\precsim^{\approx}$-relation on $\mathscr{B}[p,q]/\approx$ (two arrow-headed line segments) combines with $\leq^{\approx}$ on $\mathscr{B}[p,q]/\approx$.

Let us consider some general properties of the $\leq^+_{\precsim}$-relation on $\mathscr{B}$. First, $\leq^+_{\precsim}$ is a *preorder* of Prop($\mathscr{L}$); $\psi \leq^+_{\precsim} \varphi$ and $\varphi \leq^+_{\precsim} \psi$ obtain for every pair of propositions with $\psi \neq \varphi$ and $\psi \approx \varphi$. Second, as $\psi \leq^+ \varphi$ implies $\psi \leq^+_{\precsim} \varphi$, the latter order is *not truth-value dependent* as the former sometimes favours false theories to true ones. The $\leq^+_{\precsim}$-order is to be preferred to the $\leq^{\Delta}_{\precsim}$-order on Prop($\mathscr{L}$), which was the proposal of subsection 6.2.3. Third, $[\![\varphi]\!] \subseteq$ Prop($\mathscr{L}$) is convex regarding $\leq^+_{\precsim}$ if for all $\chi \in$ Prop($\mathscr{L}$) and all $\varphi, \varphi' \in [\![\varphi]\!]$, then $\varphi \leq^+_{\precsim} \chi \leq^+_{\precsim} \varphi'$ implies $\chi \in [\![\varphi]\!]$.

Observation 6.23: $[\![\varphi]\!]$ is convex regarding $\leq^+_{\precsim}$.

Proof: Let $\varphi \leq^+_{\precsim} \chi \leq^+_{\precsim} \varphi'$, and $\varphi \approx \varphi'$. By definition this means $\varphi' \leq^+_{\precsim} \varphi \leq^+_{\precsim} \chi \leq^+_{\precsim} \varphi' \leq^+_{\precsim} \varphi$. The transitivity of $<^+_{\precsim}$, implies that $\chi \approx \varphi$, and $\chi \approx \varphi'$. ⊠

PROPOSITION 6.24: $\leq^+_{\precsim}$ is weakly context independent on Prop($\mathscr{L}$).

Proof: Assume $\psi <^+_{\precsim} \varphi$; then $h_{\mathscr{B}\approx}([\![\psi]\!]) \leq h_{\mathscr{B}\approx}([\![\varphi]\!])$ and $h_{\mathscr{B}/I}([\psi]) \leq h_{\mathscr{B}/I}([\varphi])$. This relative height does not change under translation; therefore $h_{\mathscr{B}'/I}([\psi']) \leq h_{\mathscr{B}'/I}([\varphi'])$. We have to consider two cases: a. $h_{\mathscr{B}'/I}([\psi']) < h_{\mathscr{B}'/I}([\varphi'])$ and b.

$h_{\mathscr{B}'/\mathrm{I}}([\psi']) = h_{\mathscr{B}'/\mathrm{I}}([\varphi'])$. First, a. immediately implies $\varphi' \nless^{+}_{\precsim} \psi'$; Second, b. and $\psi <^{+}_{\precsim} \varphi$ imply $[\psi] \prec_h [\varphi]$ and since $\prec_h$ is strictly context independent this means $[\psi'] \prec_{h'} [\varphi']$ and again $\varphi' \nless^{+}_{\precsim} \psi'$. ⊠

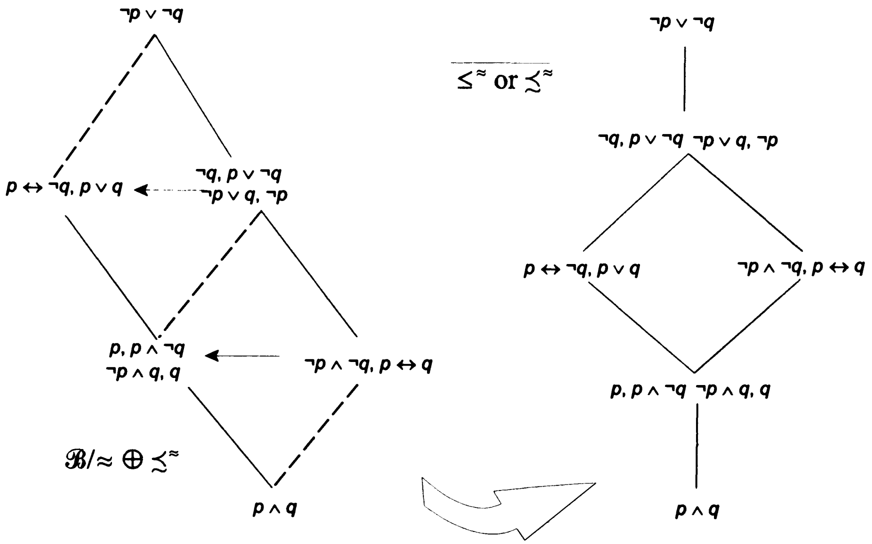


Fig. 10.

In this subsection, we saw that the representatives of $\leq^{+}$ and $\precsim_h$ on $\mathscr{B}$ are compatible. Next, the refined verisimilitude was defined to be the transitive closure of the disjunction of these two ordering relations. The resulting preorder is not truth-value dependent and is weakly context independent. Figure 1 of the appendix shows the ordering of all equivalence classes in the three propositional case (thanks are due to G.R. Renardel de Lavalette who created the diagram).

6.5. QUANTITATIVE REFINED VERISIMILITUDE: $\sqsubseteq^{+}$

Now that I have explained my comparative approach-to-the-truth proposal, the next step is to prove that this comparative definition naturally leads to a plausible quantitative refined verisimilitude definition. Although the structuralist *naive* quantitative proposal defines a content order, we saw that the *refined* quantitative version has a likeness character (Chapter 3); according to that structuralist definition not the negation of the truth, but the most distant constituent is the worst possible theory. There is a lacuna in the spectrum of the closer-to-the-truth definitions as to our knowledge, there is no *quantitative* refined content proposal; that is a quantitative verisimilitude order, according to which the negation of the truth is the worst possible theory; and that takes the distances between the constituents into account. My quantitative verisimilitude fills this void; and, to cap it all, the refined verisimilitude proposal implies my quantitative refinement.

I start with the observation that the cardinal number of the symmetric difference yields a metric (the definition of this notion can be found in Chapter 1).

PROPOSITION 6.25: Let U be some set and g: $U \times U \rightarrow \mathbb{R}$ be defined by $g(x, y) := |x \Delta y|$. Then, g is a metric on U.

Proof: It is easy to check that $g(x, y) = 0$ iff $x = y$, and $g(x, y) = g(y, x)$. Moreover, also $g(x, y) + g(y, z) \geq g(x, z)$ obtains, since $|x \Delta y| + |y \Delta z| \geq |(x \Delta z) \cup ((x \cap z) - y) \cup (y - (x \cup z))| \geq |x \Delta z|$. ⊠

The structuralist content proposal is based on the cardinality of the symmetric difference metric (Chapter 2). It is also customary to use the preceding metric to fix the distances between propositional constituents. If the truth equals $p_1 \wedge \ldots \wedge p_n$ the symmetric difference metric reduces to: $\mathrm{d}(\alpha, \tau) :=_{def} |[\mathrm{pl}(\alpha)]^c|$ since $\mathrm{pl}(\tau) = \mathrm{voc}(\mathscr{L})$. Hence, constituent β is quantitatively at least as good as α iff $\mathrm{d}(\beta, \tau) \leq \mathrm{d}(\alpha, \tau)$, which equals $|\mathrm{pl}(\beta)| \geq |\mathrm{pl}(\alpha)|$. In subsection 6.4.2 this relation has been abbreviated with $\beta \precsim_h \alpha$.

The distance between constituents enables us to define a refined distance on the propositions of $\mathscr{B}_{\mathscr{L}}$. Let us assume that $\mathscr{L}$ is a finite propositional language with the complete empirical truth $\tau := p_1 \wedge \ldots \wedge p_n$.

DEFINITION 6.13: $\mathrm{D^r}(\psi, \tau) :=_{def} \Sigma_{\beta \in \mathrm{Cnst}(\psi)} \mathrm{d}(\beta, \tau)$ defines the *refined distance* between ψ and the truth τ; ψ is quantitatively *at least as verisimilar as* φ iff $\mathrm{D^r}(\psi, \tau) \leq \mathrm{D^r}(\varphi, \tau)$
Notation: $\psi \sqsubseteq^+ \varphi$.

Obviously, $\psi \sqsubset^+ \varphi := \mathrm{D^r}(\psi, \tau) < \mathrm{D^r}(\varphi, \tau)$. $\mathrm{D^r}(\psi, \tau)$ can be normalized by dividing it by $\mathrm{D}_{max}(\psi, \tau)$. This maximal distance in $\mathscr{L}[p_1, \ldots, p_N]$ is equal to $\Sigma_{n=0}^{N} n \binom{N}{n} = N.2^{N-1}$; however, we do not have any reason to do so. Let us consider the distances of the propositions in the example of two atomic propositions.

EXAMPLE: If $p \wedge q$ is the truth of $\mathscr{L}[p,q]$, then $\mathrm{D^r}(\varphi, \tau) = 0$ iff $\tau \equiv \varphi$; $\mathrm{D^r}(\varphi, \tau) = 1$ for $\varphi \equiv p, q, p \wedge \neg q, \neg p \wedge q$; $\mathrm{D^r}(\varphi, \tau) = 2$ for $\varphi \equiv p \vee q, p \leftrightarrow q, \neg p \leftrightarrow q, \neg p \wedge \neg q$; $\mathrm{D^r}(\varphi, \tau) = 3$ for $\varphi \equiv \neg p, \neg q, p \vee \neg q, \neg p \vee q$; $\mathrm{D^r}(\varphi, \tau) = 4$ for $\varphi \equiv \neg p \vee \neg q$.
End Example

My quantitative version of refined verisimilitude differs from the quantitative structuralist proposals (see Figure 5, p. 51).

PROPOSITION 6.26: Let φ, ψ be propositions of $\mathscr{L}$; then $\psi \prec_h \varphi$ implies $\psi \sqsubset^+ \varphi$.

Proof: Let μ be a bijection between Cnst(ψ) and Cnst(φ) such that for all $\beta \in$ Cnst(ψ): $\beta \prec \mu(\beta)$; $\beta \prec \mu(\beta)$ implies $d(\beta,\tau) < d(\mu(\beta), \tau)$ (subsection 6.4.2, p. 214). Hence, $\Sigma_{\beta\in\mathrm{Cnst}(\psi-\varphi)}\, d(\beta,\tau) < \Sigma_{\mu(\beta)\in\mathrm{Cnst}(\varphi-\psi)}\, d(\mu(\beta),\tau)$ and $D^r(\psi, \tau) = \Sigma_{\beta\in\mathrm{Cnst}(\psi-\varphi)}\, d(\beta,\tau) + \Sigma_{\beta\in\mathrm{Cnst}(\psi\cap\varphi)}\, d(\beta,\tau) < \Sigma_{\mu(\beta)\in\mathrm{Cnst}(\varphi-\psi)}\, d(\mu(\beta),\tau) + \Sigma_{\beta\in\mathrm{Cnst}(\psi\cap\varphi)}\, d(\beta,\tau) = D^r(\varphi, \tau)$ ⊠

Before being able to prove the theorem promised at the start of this section, I need to prove that if the truth is complete, the elements of $[\varphi]\sim$ are at the same (refined) distance to the truth.

Lemma 27: If the truth is complete then $\psi' \sim_\tau \psi$ implies $D^r(\psi', \tau) = D^r(\psi, \tau)$.
Proof: if the truth is complete, $\psi' \sim_\tau \psi$ means by that $\mathrm{Cnst}(\psi')\, \Delta\, \mathrm{Cnst}(\psi) \equiv \tau$, and since $D^r(\tau, \tau) = 0$, it follows that $D^r(\psi', \tau) = D^r(\psi, \tau)$. ⊠

THEOREM 28: Let the truth τ be complete. Then $\psi \leq^+_{\precsim} \varphi$ implies $\psi \sqsubseteq^+ \varphi$

Proof: $[\![\psi]\!]\ \overline{\leq^{\approx} \text{or} \precsim^{\approx}}\ [\![\varphi]\!]$ iff $\exists n \in \mathbb{N}$: $[\![\psi]\!] = [\![\psi_1]\!]\ \rho\ [\![\psi_2]\!]\ \rho \ldots \rho\ [\![\psi_n]\!] = [\![\varphi]\!]$ for $\rho = \leq^{\approx}$ or $\precsim^{\approx}$. Hence, if we prove that $[\![\psi]\!]\ \rho\ [\![\varphi]\!]$ implies $\psi \sqsubseteq^+ \varphi$ then we are done. $[\![\psi]\!] \leq^{\approx}$ or $\precsim^{\approx}$ $[\![\varphi]\!]$ iff 1. $[\psi] = [\varphi]$ or 2. $[\psi'] <_{B/I} [\varphi']$ for some $[\psi']$ in $[\![\psi]\!]$, and some $[\varphi']$ in $[\![\varphi]\!]$ or 3. $[\psi] \precsim [\varphi]$. 1. The preceding lemma shows that if $[\psi] = [\varphi]$ then $D^r(\varphi, \tau) = D^r(\psi, \tau)$, and therefore $\psi \sqsubseteq^+ \varphi$. 2. Let $[\psi'] <_{\mathcal{B}/I} [\varphi']$. As all elements in the equivalence classes have the same distance to the truth, we can compare $\psi \vee \tau$ and $\varphi \vee \tau$. Further, $[\psi'] <_{\mathcal{B}/I} [\varphi']$ implies $\psi \vee \tau \models \varphi \vee \tau$, and since $\mathrm{Cnst}(\psi \vee \tau) \subset \mathrm{Cnst}(\varphi \vee \tau)$, we have $\psi \sqsubset^+ \varphi$. 3. Finally, let $[\psi] \precsim [\varphi]$. Then, $\psi \vee \tau \precsim \varphi \vee \tau$, and there is a one-one mapping μ from $\mathrm{Cnst}(\psi \vee \tau)$ onto $\mathrm{Cnst}(\varphi \vee \tau)$ such that for all $\alpha \in \mathrm{Cnst}(\psi)$, $|\mathrm{pl}(\alpha)| \geq |\mathrm{pl}(\mu(\alpha))|$. For all α, $d(\alpha,\tau) \leq d(\mu(\alpha))$ holds, implying $\Sigma_{\alpha\in\mathrm{Cnst}(\psi)} d(\alpha,\tau) \leq \Sigma_{\mu(\alpha)\in\mathrm{Cnst}(\varphi)} d(\mu(\alpha),\tau)$. Therefore $\psi \sqsubseteq^+ \varphi$. ⊠

As to the other features of the $\sqsubseteq$-proposal, it goes without saying that it establishes a linear preorder. Further, the definition is not truth-value dependent. Finally we observe that the quantitative approach-to-the-truth order fails to be weakly context independent.

Observation 6.28: $\sqsubset^+$ is *not even weakly context independent.*
Proof: A counterexample suffices to prove the observation. The idea behind the counterexample is that in the original language the sum of numerous small distances between ψ and the truth increases more rapidly when the language increases, than the distance between the truth and one distant constituent. Consider $\mathcal{L}[p_1, p_2, p_3]$ with the truth $\tau := p_1 \wedge p_2 \wedge p_3$, and theories $\psi := (p_1 \wedge p_2 \wedge \neg p_3) \vee (p_1 \wedge \neg p_2 \wedge p_3)$ and $\varphi := \neg p_1 \wedge \neg p_2 \wedge \neg p_3$; $d(\psi,\tau) = 2 < 3 = d(\varphi,\tau)$. Let us expand the language into $\mathcal{L}'[p_1, \ldots, p_6]$ with the truth $p_1 \wedge \ldots \wedge p_6$. Then $d(\psi', \tau')$ equals $2 \times ((1\times8)+12) = 40 > 36 = 1 \times ((3\times8) + 12)$ which is again equal to $d(\varphi',\tau')$. Thus $\psi \sqsubset^+ \varphi$, and $\varphi' \sqsubset^+ \psi'$. ⊠

The counterexample shows that although ψ' is "less false" than φ', the definition punishes its lack of logical strength.

At the end of this section, I want to answer an objection of Niiniluoto against my approach-to-the-truth proposal, which reads thus. Imagine a situation in which we are outside and must guess the number of people in a house. Then it is plausible to claim that the guess of exactly 50 is better than exactly 100 if there is only one person. In Niiniluoto's approach, all sentences p_0 := "there are no persons in the house", p_1 := "there is exactly one person," p_2 := "there are exactly two persons" etc. are complete potential answers to the cognitive problem: "How many people are there in the house?" They are treated in the same way as the constituent is in the finite propositional case. Then Niiniluoto suggests that the logical consequences of p_{100} do not matter much and this complete answer is the worst possible answer. Likeness intuitions back up the idea that p_{50} is better than p_{100}, but why would $\neg p_1$ be better than p_{100}? It is doubtful whether Niiniluoto's numerical guess example shows that an overall likeness approach is to be preferred to a refined verisimilitude proposal like mine.

Without plunging into a never ending discussion about all intuitions pro and contra, obviously my quantitative approach-to-the-truth order of the present section is perfectly able to handle quantitative examples similar to the guess about the number of people in the house. The $p_0 \ldots p_{100}$ order may be embedded into my linear quantitative order. More generally, it remains to be seen whether the constituent analysis is the most appropriate approach for strict quantitative examples like the one preceding.

6.6. SUMMARY AND PROSPECTS

Let me recapitulate the most important facts about refined verisimilitude and sketch the prospects for future research. At the beginning of this chapter I quoted Niiniluoto explaining the drawback of all—in my terms—verisimilitude proposals.[21] They "don't pay any attention to the underlying metric structure of the space of complete answers". Miller formulates the same issue in terms of "horizontal" and "vertical improvement". In the preceding sections we saw, however, that this feature of verisimilitude definitions is mendable. My three approach-to-the-truth proposals all take the topology of constituents into account. Regarding a Lindenbaum algebra, I merged "horizontal" and "vertical" improvements.

6.6.1. *Summary*

1. In this chapter we assumed $\mathcal{L}$ to be a finite propositional language.

2. I proved the compatibility of the vertical $\leq^{\Delta}$-order and the horizontal $\preceq_{h}$-order (for propositions of the same logical strength) (prop 6.6, p.199). The second is based on the $\leq^{\Delta}$-order for constituents (prop. 6.2, p.197).
3. Next, I merged the $\leq^{\Delta}$-order and the likeness order using the transitive closure of the disjunction of the ordering relations. For all $\psi, \varphi \in \mathrm{Prop}(\mathscr{L})$: $\psi \leq^{\Delta} \varphi \Rightarrow \psi \leq^{\Delta}_{\precsim} \varphi$ (Section 6.2) The resulting order is truth-value dependent;
4. I proved that the $\leq^{+}$-order is homomorphic with the Lindenbaum algebra of language $\mathscr{L}$.
5. Consequently, my second proposal took the stronger content ordering $\leq^{+}$ as its vertical component and proved it to be compatible with a strengthening of the horizontal likeness order, $\precsim$;
6. Subsequently, I introduced my *refined verisimilitude* definition by merging $\leq^{+}$ and $\precsim$. The following are the properties of resulting ordering $\leq^{+}_{\precsim}$.
   a. It incorporates the constituent topology.
   b. It systematically applies this topology to weaker propositions.
   c. it is not truth-value dependent.
   d. It favours all propositions to the negation truth.
   e. It favours the theory with more true consequences to the one with less.
   f. It is open to a modal version that might block the child's-play objection.
7. The refined verisimilitude ordering leads naturally to quantitative refined verisimilitude of the last section $\sqsubseteq^{+}$.
8. Finally I showed that the Δ-definition, the +-definition, my refined verisimilitude, and its quantitative version together form a linear ordering of increasing logical strength: $\psi \leq^{\Delta} \varphi \Rightarrow \psi \leq^{+} \varphi \Rightarrow \psi \leq^{+}_{\precsim} \varphi \Rightarrow \psi \sqsubseteq^{+} \varphi$ (Sections 6.3 and 6.4). Textbox 5 illustrates the relation between the $\leq^{\Delta}_{\precsim}$-order and this linear ordering

$$\begin{tikzcd} & \psi \leq^{+} \varphi \arrow[rr] & & \psi \leq^{+}_{\precsim} \varphi \arrow[r] & \psi \sqsubseteq^{+} \varphi \\ \psi \leq^{\Delta} \varphi \arrow[ur] \arrow[rr] & & \psi \leq^{\Delta}_{\precsim} \varphi \arrow[ur] & & \end{tikzcd}$$

Textbox 4. The logical relations between the various content orderings.

Figure 10 at the end of this section illustrates the two lines of investigation using the two propositional example.

6.6.2. *Prospects*

Finally, let us turn towards the prospect for future research. In the first place, I have already mentioned that it is necessary to develop a modal version of the refined verisimilitude definition. Parallel to the structuralist case, the modal version will definitely block the child's-play argument.

Secondly, it is necessary to extend the set of possible mathematical application. To begin with, we have to answer the question: how are we to define refined verisimilitude *if the truth is incomplete*? Part of this answer reads: we should use the likeness refinement of a conservative restriction of $\mathscr{L}$, which has a complete truth. The next obvious generalization concerns the size of the language. How are we to define the $\leq^{+}_{\lesssim}$-order if $\mathscr{L}$ is an infinite propositional language where the truth is not axiomatizable? If the truth is complete, perhaps we may order the ultrafilters of the Lindenbaum algebra to establish the likeness refinement. Furthermore, we need to adapt the refined verisimilitude definition to first order logic. As a preliminary we may consider monadic predicate languages; and then, perhaps we have to adjust our refined verisimilitude proposal to cylindric algebras. Since we do not have adequate answers to these questions of expansion, we are unable to estimate to what extent refined verisimilitude hinges on finite propositional languages.

A third important expansion of my refined verisimilitude research concerns the question of how it contributes to a plausible answer to the epistemic question of approach-to-the-truth. This leads us to the final and most important future project, which concerns the relation between approach-to-the-truth research and all kinds of logical investigations where preference relations play an important role. Of course, there is already a connection between non-monotonic reasoning and truthlikeness via Hilpinen's approximate logic, counterfactuals to conditional logic. There is, however, a need for a more general understanding of the relation between approach to the truth, verisimilitude and truthlikeness, on one side, and non-monotonic logic and belief revision on the other. Both realms of investigation would benefit from such exploration.

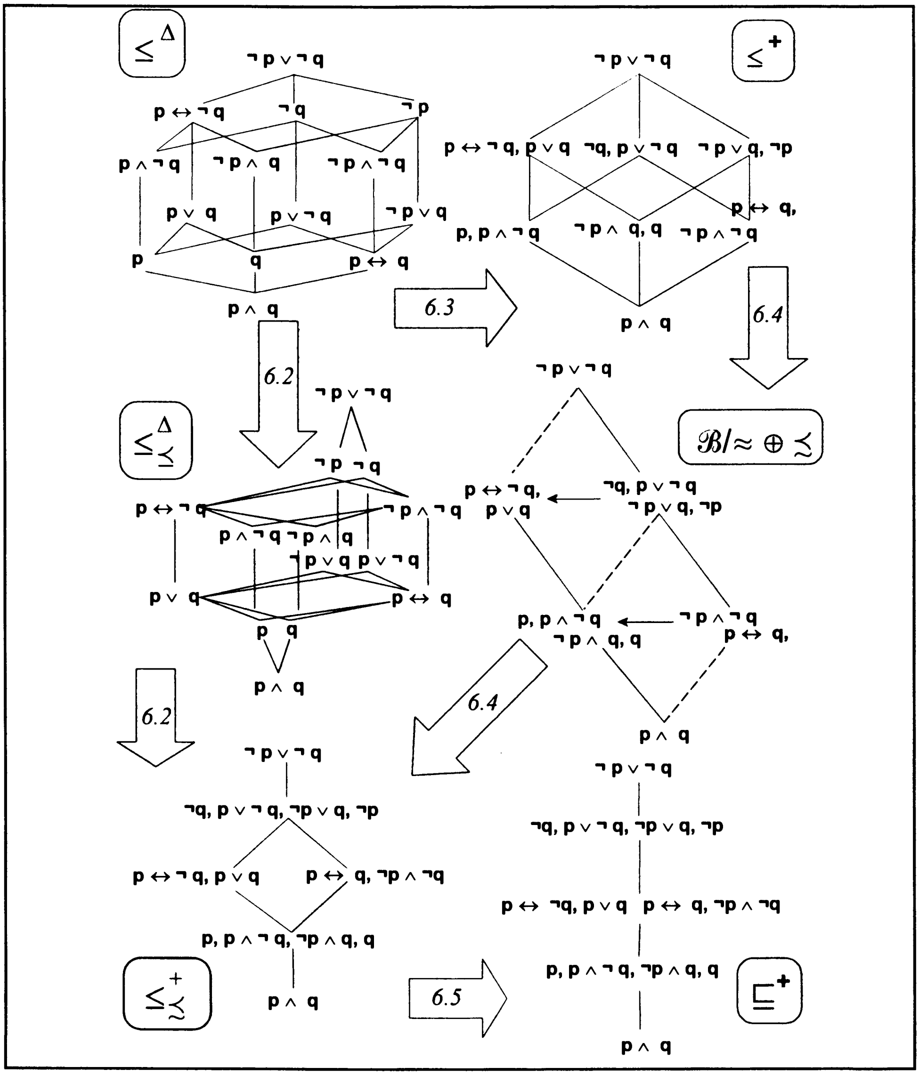


Fig. 11. Outline of Chapter 6 using the ordering of two-proposition example.

APPENDIX

7.1. Independent truth- and falsity-content (ad p. 11)

Popper's assumptions about the truth-content and falsity-content imply that increase of the true part of a theory is the same as increase of its false part. As a consequence, it is impossible to improve a false theory according to Popper's original proposal. We can easily correct Popper's error by dropping the assumption that the union of the truth- and falsity-content of a theory is deductively closed; then nothing blocks an independent comparison of the truth- and falsity-contents.

Let $\mathcal{T}$ be a scientific theory, the consequences of which may be represented by propositions of a formal language $\mathcal{L}$; and let τ be the strongest truth of $\mathcal{L}$. Moreover, let $\varphi_t \in \mathrm{Cn}(\tau)$ axiomatize the truth-content of $\mathcal{T}$, $Ct_T(\varphi_t)$, and let φ_f determine the falsity-content of $\mathcal{T}$, $Ct_F(\varphi) = \mathrm{Cn}(\varphi_f) \cap \mathrm{An}(\neg\tau)$. Notice, that φ_t need not be equivalent to $\varphi_f \vee \tau$ as is the case in Popper's approach. Then, $T(\varphi_t, \varphi_f) := Ct_T(\varphi_t) \cup Ct_F(\varphi_f)$ represents the true and false consequences of $\mathcal{T}$. Note that we do not assume that $T(\varphi_t, \varphi_f) = \mathrm{Cn}(\varphi_t \wedge \varphi_f)$. Additionally, we shall assume that $Ct_T(\varphi_t) \cup Ct_F(\varphi_f)$ is consistent. Now, we may adjust Popper's definition as follows: A theory $T(\psi_t, \psi_f)$ is at least as verisimilar as $T(\varphi_t, \varphi_f)$ iff a. $Ct_T(\varphi_t) \subseteq Ct_T(\psi_t)$ b. $Ct_F(\psi_f) \subseteq Ct_F(\varphi_f)$. My representation of $\mathcal{T}$ is not deductively closed. This is rather unconventional, but saves Popper's intuitions since the truth-content of theories, and their falsity-content can be compared independently. Interestingly, sometimes, as a result of the falsity clause, $\leq_\tau^{P'}$ favours logical weakness to logical strength. For instance, regarding the propositional language $\mathcal{L}[p, q]$, we obtain the following ordering $T(\neg p \vee q, \neg p \vee \neg q) <_\tau^{P'} T(\neg p \vee q, \neg p)$. It is fair to concede that the present revision of Popper's proposal is at most a theoretical one. In practice, scientific theories are considered to be deductively closed. A possibly less ad hoc way to reestablish Popper's definition is to model theories using relevance logic instead of classical logic (Schurz and Weingartner 1987).

7.2. Niiniluoto's ordering for various values of γ and γ' (ad page 85)

Likeness table

$\gamma = 10\gamma'$		$\gamma = 2\gamma'$		$\gamma = \gamma'$	
$p \wedge q$	0	$p \wedge q$	0	$p \wedge q$	0
p, q	$0.25\gamma'$	p, q	$0.25\gamma'$	p, q	$0.25\gamma'$
$p \vee q, p \leftrightarrow q$	$0.5\gamma'$	$p \vee q, p \leftrightarrow q$	$0.5\gamma'$	$p \vee q, p \leftrightarrow q$	$0.5\gamma'$
$p \rightarrow q, q \rightarrow p$	$0.75\gamma'$	$p \rightarrow q, q \rightarrow p$	$0.75\gamma'$	$p \rightarrow q, q \rightarrow p,$ $p \wedge \neg q, \neg p \wedge q$	$0.75\gamma'$
taut	$1.0\gamma'$	taut	$1.0\gamma'$		
$p \wedge \neg q, \neg p \wedge q$	$5.25\gamma'$	$p \wedge \neg q, \neg p \wedge q$	$1.25\gamma'$		
$\neg p \leftrightarrow q$	$5.5\gamma'$	$\neg p \leftrightarrow q$	$1.5\gamma'$	$\neg p \leftrightarrow q$, taut	$1.0\gamma'$
$\neg p, \neg q$	$5.25\gamma'$	$\neg p, \neg q$	$1.75\gamma'$	$\neg p, \neg q$	$1.25\gamma'$
$\neg p \vee \neg q$	$6.0\gamma'$	$\neg p \vee \neg q$	$2.0\gamma'$	$\neg p \wedge \neg q$	$1.5\gamma'$
$\neg p \wedge \neg q$	$10.5\gamma'$	$\neg p \wedge \neg q$	$2.5\gamma'$	$\neg p \vee \neg q$	$1.5\gamma'$

Content table

$2\gamma = \gamma'$		$3\gamma = \gamma'$		$8\gamma = \gamma'$	
$p \wedge q$	0	$p \wedge q$	0	$p \wedge q$	0
p, q	$0.25\gamma'$	p, q	$0.25\gamma'$	p, q	$0.25\gamma'$
$p \vee q, p \leftrightarrow q$ $p \wedge \neg q, \neg p \wedge q$	$0.5\gamma'$	$p \vee q, p \leftrightarrow q$	$0.5\gamma'$	$p \wedge \neg q, \neg p \wedge q$	$0.31\gamma'$
		$p \wedge \neg q, \neg p \wedge q$	$0.58\gamma'$	$p \vee q, p \leftrightarrow q$	$0.5\gamma'$
$p \rightarrow q, q \rightarrow p$ $\neg p \leftrightarrow q$	$0.75\gamma'$	$\neg p \leftrightarrow q$	$0.67\gamma'$	$\neg p \leftrightarrow q$	$0.56\gamma'$
		$p \rightarrow q, q \rightarrow p$	$0.75\gamma'$	$\neg p \wedge \neg q$	$0.63\gamma'$
taut $\neg p, \neg q$ $\neg p \wedge \neg q$	$1.0\ \gamma'$	$\neg p \wedge \neg q$	$0.83\gamma'$	$p \rightarrow q, q \rightarrow p$	$0.75\gamma'$
		$\neg p, \neg q$	$0.92\gamma'$	$\neg p, \neg q$	$0.81\gamma'$
		taut	γ'	taut	γ'
$\neg p \vee \neg q$	$1.25\gamma'$	$\neg p \vee \neg q$	$1.17\gamma'$	$\neg p \vee \neg q$	$1.06\gamma'$

7.3. The proof of subsection 3.4.3 (ad p.104).

Let $\vartriangle X := \{w \in W \mid \exists x \in X\colon w \geq x\}$, and $\triangledown X := \{w \in W \mid \exists x \in X\colon w \leq x\}$; $\max(X) := \{x \in X \mid \nexists y \in X\colon x < y\}$, and $\min(X) := \{x \in X \mid \nexists y \in X\colon y < x\}$, and $\vartriangle\min(X) := \{w \in W \mid \exists x \in \min(X)\colon w \geq x\}$, $\triangledown\max(X) := \{w \in W \mid \exists x \in \max(X)\colon w \leq x\}$.

PROPOSITION 7.1: $\vartriangle X = \vartriangle\min(X)$, and $\triangledown X = \triangledown\max(X)$.

Proof: $\Rightarrow$ Let $w \in \vartriangle X$. Then $\exists x \in X$ such that $w \geq x$, and if also $x \in \min(X)$, then $w \in \vartriangle\min(X)$. If $x \notin \min(X)$, then $\exists x' \in \min(X)\colon x > x'$ and $w \geq x'$. Hence, again $w \in \vartriangle\min(X)$. $\Leftarrow$ Since $\min(X) \subseteq X$, if w $\in \vartriangle\min(X)$, then w $\in \vartriangle X$. ⊠

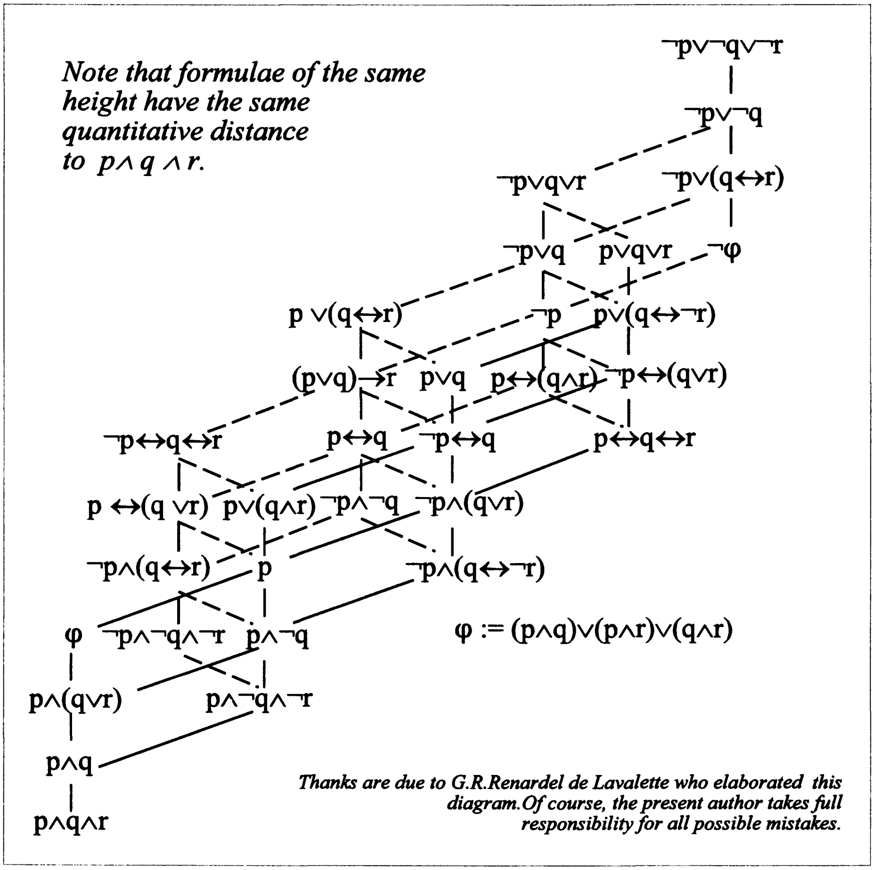


Fig. 1. The (refined) verisimilitude ordering of all 3 propositional sentences

NOTES

CHAPTER 1

1. Peirce (1965) fifth volume section 565.
2. See for instance Niiniluoto (1984), p.76-77.
3. Whether the scientific claims about these different fields are, or should be, reducible is still a question of debate.
4. Laudan (1981).
5. Kuipers (1987), p.7.
6. $\mathscr{L}$ also relates to the cognitive problem that is to be solved. I elaborate on this in Chapter 5.
7. In the sequel, the term "proposition φ of a language $\mathscr{L}$" ($\varphi \in \mathrm{Sent}(\mathscr{L})$) refers to the set of equivalent $\mathscr{L}$-sentences, $[\varphi]_{\equiv}$, and has nothing to do with *intentional objects*. Often, we refer to such a set by one representative of it, and occasionally will use proposition, sentence and statements almost as synonyms. In the same vain, we use the logical connectives of propositions. The set of $\mathscr{L}$ propositions is designated by $\mathrm{Prop}(\mathscr{L}) := \{[\varphi]_{\equiv} \mid \varphi \in \mathrm{Sent}(\mathscr{L}))\}$.
8. Niiniluoto (1987), p.256.
9. The empirical sentences are the non-tautological, and non-contradictory sentences of a language.
10. Popper (1963), p 234.
11. Popper (1963), p.233.
12. Rescher (1967, chap. vi) gives a nice introduction to Kleene's three-valued logic.
13. Weston (1992).
14. See Kuipers (1987), p.88. Not surprisingly, the explication of such a general idea of progress has a wider scope than only Popper's methodology. There are also implications for instrumentalistic and technological progress.
15. Popper (1963), p.233.
16. *Ibid.*
17. Popper (1963), chapter X, section xi.
18. Compare with the *dual consequences* of Jan Woleński (1990), p. 619.
19. The Miller-Tichý result has also consequence for followers of Popper. For example, Lakatos (1970), p.116 "follows Popper" and defines "sophisticated falsification". According to Lakatos's proposal T is falsified if there new theory T'

such that there is a true novel fact e, such that $T' \vdash e \vdash \neg T$, and $T' \vdash T \vee \tau$ (τ is the truth); consequence is that T can only be falsified using a *true* theory T' (else $T' = \perp$); which is at least improbable.
20. Hattiangadi (1983), note 10; Miller (1974).
21. See Tichý (1974) and Miller (1974).
22. $<\psi, \chi>$ has a intermediate if $\exists\varphi \in \mathrm{Sent}(\mathscr{L})$: $\psi <^{P}_{p \wedge q} \varphi <^{P}_{p \wedge q} \chi$.
23. See Tichý (1974), p.157, note 2; Harris (1974), p.165; Miller (1978), p.415 and Niiniluoto (1987), p. 190.
24. McCarthy (1980) on circumscription was an important starting point of non-monotonic reasoning, see also Makinson (1994). Brewka, Dix and Kolonige (1997) is a comprehensive and readable introduction on non-monotonic reasoning.
25. There is an increasing interest in the relation between verisimilitude and standard concepts in philosophical logic. See e.g. van Benthem (1987), Ryan and Schobbens (1995).
26. Schurz and Weingartner's (1987) revision of Popper's definition drops the assumption that all consequences of theories are equally important. The definition only takes the "relevant consequences" of theories into account. According to their analysis, however, one consequence φ may be relevant whereas an equivalent reformulation of φ is not. For instance, $(\neg p \wedge \neg q) \rightarrow \neg p$ is a relevant implication, and it is equivalent to the irrelevant $p \rightarrow (p \vee q)$ (see Schurz and Weingartner (1987, p.54)).
27. Some of the definitions presented in Chapter 3–4 are to be found in Kuipers (1987); among the most recent proposals are Woleński (1990), Gerla (1992), Zamora Bonilla (1992/6), Volpe (1995), and Kieseppä (1996). Niiniluoto (1998) gives an excellent overview of "the third period" in the approach-to-the-truth research.
28. E.g. see van Benthem (1987).
29. Although Popper used the content and similarity notions in his original presentation, Hilpinen (1976, p. 38) stressed the difference between the content and likeness approach. Finally, it was Oddie who has claimed that the content and likeness distinction obtains for almost all definitions. See e.g. Oddie (1990).
30. Niiniluoto (1987), p.459.
31. Popper (1963), p.391-398.
32. Popper (1972), p. 56.
33. See also footnote 24[a] in Popper (1972), p. 56.
34. Miller (1978).
35. Oddie (1986), sect.1.3.
36. See Hilpinen (1975).
37. From Niiniluoto (1987b), p. 13-15.
38. Kuipers (1992), p.317-319.
39. Kuipers (1992b) applies the idealization concretization ideas of Nowak (1980).
40. Brink and Heidema (1987).

41. For a more elaborate introduction see the Section 2.6.
42. See e.g. Suppe (1977), the introduction.
43. The example is also to be found in Oddie (1987).
44. In the next chapter, we shall show that if the language $\mathscr{L}$ will be extended into a modal language $\mathscr{L}_{S5}$, then Popper's revision can be interpreted as "more correct permissions and prohibitions".
45. Of course, weak theories do not need to imply literals at all. Real likeness definitions are all equipped with technical devices that also measure the distance to the truth of weak theories.
46. Miller (1974b).
47. See Miller (1978) and Kuipers (1992), respectively.
48. Popper (1963), p.233, first paragraph.
49. Tichý (1974), p158, p.159.
50. Oddie (1986), p 13.
51. Hilpinen (1976), p.38.
52. The complete falsehood notion is not restricted to propositional languages. Regarding more sophisticated languages the complete falsehood is the proposition that contradicts the truth in all restrictions of the language.
53. Tichý (1974, p.157 note 2).
54. The reason for this phenomenon will become clear in Chapter 2.
55. In Miller (1994) the author has changed his mind and dismisses the argument.
56. The Chapter 2 shows that it is the *descriptive* nature of the content approaches that causes the trouble; our modal content proposal blocks the child's-play argument.
57. Our analysis seems to agree with that of Miller (1994).
58. Although Miller uses three propositions, two suffice to formulate the argument.
59. Neither does Miller's (1975) argument against quantitative estimation.
60. Miller (1978, p. 431, last paragraph) clearly uses this meta-interpretation.
61. For numerous other examples see Niiniluoto (1987), chapter 1.
62. Niiniluoto (1987), p.2.
63. Miller (1994), p.207-208.
64. Niiniluoto (1987) sections 6.4 and 6.5 present and assess quite a number.
65. See also Niiniluoto (1987), p.233 (M10).
66. The term is Roberto Festa's, see Kuipers (1987a), p.85.
67. Miller (1976) and Niiniluoto (1987), section 6.2.
68. E.g. Chang and Keisler (1973), p.19.
69. See van Benthem (1996).
70. Kuipers (1982), p.352-353.
71. Niiniluoto (1987), p.380-382.
72. Niiniluoto (1987), sec. 6.8.

CHAPTER 2

1. Miller (1974) and Tichý (1974).
2. Miller (1978, p. 429).
3. This is in accordance with Miller (1994).
4. *Ibid* p. 416.
5. Miller (1978, p.423).
6. Note that Miller calls here *A+B*, what he calls *A.B* in Miller (1974a) and vice versa.
7. Miller (1978) p. 424. Further, the Brouwerian algebra is also known as the dual of the Heyting algebra. Curry calls it the subtractive lattice, and Popper the dual-intuitionistic calculus.
8. Miller (1974) p.168.
9. *Ibid.* p.169 note 1.
10. Miller (1978, p.427).
11. Miller (1974, p.174) second corollary of Theorem 5.
12. *Ibid.* Lemma on p.173.
13. *Ibid.* p.427.
14. Obviously, $\psi \leftrightarrow \neg\tau \vDash \varphi \leftrightarrow \neg\tau$ equals $(\psi \wedge \neg\tau) \vee (\neg\psi \wedge \tau) \vDash (\varphi \wedge \neg\tau) \vee (\neg\varphi \wedge \tau)$. Consequently, $[\mathrm{Mod}(\psi \wedge \neg\tau) \cup \mathrm{Mod}(\neg\psi \wedge \tau)] \subseteq [\mathrm{Mod}(\varphi \wedge \neg\tau) \cup \mathrm{Mod}(\neg\varphi \wedge \tau)]$, which is abbreviated by: $\mathrm{Mod}(\psi)\ \Delta\ \mathrm{Mod}(\tau) \subseteq \mathrm{Mod}(\varphi)\ \Delta\ \mathrm{Mod}(\tau)]$.
15. See for more detailed information Bell and Slomson 1969.
16. However, the theory of densely ordered sets without end-points is complete and first order axiomatizable (see Bell and Slomson 1969 chapter 9 section 5).
17. See also Kuipers (2000).
18. The differences between the logicistic and structuralist theory representation emerge while using more sophisticated theory representations.
19. Kuipers (1992), p.300-301.
20. *Ibid.* p.301.
21. *Ibid.* p.301. and Kuipers (1987) p.82.
22. Cf. Kuipers (1982), p.347.
23. *Ibid.* section 2.
24. Kuipers (1987a), p.82.
25. *Ibid.* p.83. Compare with the "potential falsifiers" in Popper (1963), p.385.
26. See for Kuipers's version Kuipers (1982) and Kuipers (1992).
27. Kuipers (1982, p.347) gives the same list of the possible combinations of weak and strong, theoretical and descriptive truth.
28. Note that according to Popper (1963, fifth edition, p.232, and Addendum 1) the empirical content of a theory is proportional to its logical content.

29. Niiniluoto's remarks (1987, p.381-382) seem to be inspired by the same observations.
30. See Kuipers (1982), p.353 and (1992), p.304.
31. Kuipers (1992), p.314.
32. See Miller (1978, last page) for the same assessment.
33. See for more properties of the Δ-definition van Benthem (1987).
34. Kuipers (1982), p.357. Although Kuipers (1992) refers to worlds on p. 301, the paper does not mention the incompleteness of the truth.
35. Niiniluoto (1987), p.381.
36. Kuipers (1982), the second section.
37. Although it cannot be denied that the descriptive theory of plate tectonics has been a big step in the right direction.
38. Niiniluoto (1987), p.381-382.
39. Actually, Niiniluoto proposes to replace first order constituents $\bigwedge_{j\in \mathbf{CT}i}(\exists x)Q_j(x) \wedge (x)\,[\bigvee_{j\in \mathbf{CT}i} Q_j(x)]$, by nomic constituents $\bigwedge_{j\in \mathbf{CT}i} \Diamond(\exists x)Q_j(x) \wedge \Box(x)[\bigvee_{j\in \mathbf{CT}i} Q_j(x)]$.
40. Cohen (1980) and (1987).
41. Hughes and Cresswell (1984), p.7.
42. At least according to Kuipers (1982).
43. Here again we may restrict theories to be elements of SProp($\mathscr{L}_{S5}$), but we leave the details to the reader.
44. In $\mathscr{L}_{S5}[p, q]$ with $\tau = \Box(p \wedge q)$ the complete falsehood ξ is the most distinct constituent: $\Box(\neg p \vee \neg q) \wedge \Diamond(\neg p \wedge q) \wedge \Diamond(p \wedge \neg q) \wedge (\neg p \wedge \neg q)$. More generally, it is the proposition that denies all positive and negative accessibility claims of the truth.

CHAPTER 3

1. Interestingly, Niiniluoto and Kuipers also attended this conference.
2. Hilpinen (1976) is Hilpinen's contribution to the Warsaw conference for Formal Methodology in the Methodology of the Empirical Sciences, June 17-21, 1974.
3. Oddie (1986).
4. Brink and Heidema (1987), and Burger and Heidema (1994).
5. *Ibid* p.25
6. See Chapter 5. For instance, if the characteristics C_i are p, and q, then $\neg p \wedge q$ is more similar to $p \wedge q$ then $\neg p \wedge \neg q$; however, this similarity ordering reverses if the characteristics are p and $p \leftrightarrow q$.
7. Note that both $E_u(P) \subseteq E_u(Q) \vee E_u(P) \supseteq E_u(Q)$, and $I_u(Q) \subseteq I_u(P) \vee I_u(Q) \supseteq I_u(P)$ hold.
8. See also Oddie (1990). If $\min_u(\varphi) := \{K \in E_u(\varphi)\text{: } K \supseteq K' \text{ for } \forall K' \in E_u(P)\}$, and $\max_u(\varphi) := \{K \in I_u(\varphi)\text{: } K \subseteq K' \text{ for } \forall K' \in I_u(P)\}$ then it holds that $\psi \leq^H_u \varphi$ iff

$\min_u(\psi) \subseteq \min_u(\varphi)$ and $\max_u(\psi) \subseteq \max_u(\varphi)$.
9. See also Niiniluoto (1987), p. 233.
10. For the more general case see appendix.
11. See Niiniluoto (1987), p.126.
12. Niiniluoto (1987), p.311.
13. Cf. W. K. Clifford in *Proceedings of the Manchester Literary and Philosophical Society* (February 1877, vol. xvi).
14. If $M_1 \ldots M_k$ are the atomic predicates of a monadic language $\mathscr{L}_N^k$ then a Q-predicate is defined by $Q(x) := (\pm)M_1(x) \wedge \ldots (\pm)M_k(x)$, in which $(\pm)M_i$ means that M_i is affirmed or negated.
15. Niiniluoto (1987), p. 212. It is similar to what Oddie (1990) calls the averaging method.
16. $\Delta(h_i,g) = \text{red}(\langle \Delta_{ij} | \ j \in \mathbf{I}_g \rangle)$ means that the value of the function Δ for the argument h_i and g equals the function value of red for the arguments $\langle \Delta_{i1}, \ldots, \Delta_{ij}, \ldots, \Delta_{in} \rangle$ for $j \in \mathbf{I}_g$.
17. *Ibid.* p.212
18. *Ibid* p.217
19. *Ibid.* p.214-216
20. The diagram is a copy of Fig.5 of Niiniluoto (1987), p.214.
21. See Niiniluoto (1987), p. 25 and Czekanowski (1909).
22. Sokal (1961).
23. See Niiniluoto (1987) p. 232-234.
24. *Ibid.* p.248.
25. Niiniluoto (1987) sections 6.7 and 6.8, respectively.
26. The numbers between parenthesis refer to the sections in Niiniluoto (1987) describing the relevant distance functions.
27. Niiniluoto (1987) section 2.2 and section 3.2; Niiniluoto (1987b), p.14.
28. Niiniluoto (1987), section 6.2, p. 207-209.
29. Perhaps the choice in favour of the Clifford measure has to do with the expected verisimilitude. We will come to this in the next chapter. Niiniluoto suggests that his Jyväskylä measure would produce nice results. This measure is a weighted, quantitative version of the Clifford measure. Niiniluoto (1987), p.313-320]. Although this measure would somewhat qualify the content character of the present example, it does not wipe out its overall content character.
30. Oddie (1987 a), p.37.
31. Tichý and Oddie define their truthlikeness ordering for higher order, intensional interpreted languages.
32. E.g. see Oddie (1987a), p. 30-34 or Oddie (1986), p. 60-63.
33. Oddie (1986), p.62.
34. *Ibid.* p. 126-127.
35. L.J. Cohen (1980).
36. *ibid* p.34-35.

37. Miller (1975).
38. Oddie (1987a).
39. Niiniluoto (1987), p.324. Section 9.3 gives a good analysis of difference between the Tichý-Oddie measure and the Clifford measure.
40. Oddie (1990).
41. The proof can be found in the Appendix.
42. *Ibid.* p.132.
43. In fact, it is easy to see that for all sentences φ, ψ of $\mathscr{L}[p,q]$ obtains:

$$\psi \leq_{\tau}^{T\&O} \varphi \Leftrightarrow \psi \leq_{\tau}^{Bu\&H} \varphi$$

44. Kuipers (1987), van Benthem (1987) and Kuipers (1992).
45. The relation between ideal gas models and Van der Waals gas models is an example of concretization; Kuipers (1992), section 4.
46. In the latest version of definition the t must be an element of $T\text{-}X$ instead of T.
47. Kieseppä (1995, p.184) makes related observations, although he considers the case in which there are no incomparable elements and in which the truth is complete.
48. The notation of the definition and those which are to come suggest that the relevant sets are at most denumerable. Adaptation to continuous sets does not seem to produce any substantial difficulties.
49. Kuipers (1992), p. 332-333. It equals Niiniluoto's minsum measure $\delta_{ms}(X,Y)$, (1987), p. 246.
50. *Ibid.* p.332.
51. *Ibid.* section 3.7.
52. Niiniluoto (1987), p. 247.
53. Kuipers (1992), p.319.
54. Kieseppä (1995), p.180 and Kuipers (1997), p. 161.
55. Private correspondence with Kieseppä.
56. Kuipers (1992), p.333.

CHAPTER 4

1. Note the subtle difference. The content approach considers the true consequences of the theories; the likeness rule is based on the true consequences of all non-falsified constituents.
2. Kuipers (1992), p.326.
3. Irrevisable and irreversible rules may be defined as follows. Let ρ be a rule of theory-choice; let $<_e^\rho$ be its preference ordering based on evidence e, Then ρ is

irrevisable iff $\psi <_e^\rho \varphi$ implies $\forall e' : e' \vDash e$ and $\psi <_{e'}^\rho \varphi$

irreversible iff $\psi <_e^\rho \varphi$ implies $\forall e' : e' \vDash e$ and $\varphi \nless_{e'}^\rho \psi$

Note that an irrevisable rule is functional, and a functional rule is irreversible.

4. Niiniluoto (1987), p.482. note 5.
5. Niiniluoto's point of view regarding Laudan's challenge can be found in Niiniluoto (1997).
6. *Ibid.* sect.6.8.
7. See Niiniluoto (1997).
8. Popper (1963), p.234.
9. Niiniluoto (1987), p.264-265.
10. Popper (1963), Appendix *IX, p.288.
11. Popper (1963), p.235.
12. Every reader, knowing more than the barest outlines of falsificationism, knows that my presentation is grossly simplified. For example, Lakatos (1970) sketches a gamut of falsificationist positions entirely neglected in our exposition. Taking Lakatos's idea of scientists seriously using systematic strategies to refute theories, van Benthem suggests an interesting line of further research linking belief-revision and approach to the truth. We could classify the various revision strategies in the light of approach to the truth.
13. This procedure is well illustrated by a comparison of Popper. He compared the search for truth to the search for a "black cat in a dark room that might not even be there", with thanks to A. Keupink who drew my attention to this fact.
14. The elements of I_X can also be interpreted as partial models that together constitute one big model of reality.
15. Kuipers (1996, p.86) calls $S(t)$ *a general fact* since "scientists use to speak about a general fact."
16. Kuipers (1987), p.96.
17. Kuipers (1995), p.365.
18. See Kuipers (1992), p.325.
19. Kuipers (1992), p.326.
20. See Kuipers (1998).
21. Kuipers (1992), p.308, last paragraph.
22. $R(t) \subseteq I$ trivially obtains, because $R(t)$ are situations in reality that are part of the intended applications.
23. Kuipers (1992), p. 301.
24. Kuipers (1992), p. 309.
25. See for similar objections Niiniluoto (1987), p.381.
26. Recall that if we consider a new R' and S' increase of R and decrease of S both represent increase of logical strength.
27. Kuipers (1996, p.87, p.97) changed his terminology but the principle remained the same. He uses the term "general facts" instead of "the strongest accepted law until time t", and $S(t)$ is substituted by ES(M, t): the explanatory successes of theory M at time t. It contains I as a subset. He gives "corrected versions of the laws of Galileo and Kepler" as examples of the general test implications that are supposed to be true.

28. As mentioned earlier, substitution of the average *sum* of the minimal distances for the *sum* of those distances would balance the situation.
29. Kuipers (1982), p. 357.
30. For more instances see Niiniluoto (1987), p.2-18.
31. Niiniluoto presents his answer to the epistemic problem in (1987), chap 7.
32. *Ibid.* p. 269.
33. *Ibid* p. 270.
34. Niiniluoto (1987), the seventh note of chapter 7.
35. See Niiniluoto (1987), p.270.
36. Niiniluoto (1987), p.274 formula (19).
37. *Ibid.* Result (IV) of equation (18).
38. *Ibid* p.273
39. Niiniluoto (1987) , p.275.
40. For the information of the foregoing paragraph, see Niiniluoto (1987), p. 275-275.
41. *Ibid.* p. 268-269.
42. As Carnap (1962, p.523) explains, we could have made another choice. The mode or median are also candidates. The latter possibilities, however, are less satisfactory.
43. Niiniluoto (1987), p.269.
44. *Ibid.* p.269, 270.
45. Carnap (1962, chap IX) warns for this kind of problem.
46. Here $\mathrm{ver}(h_3 / e) = 1 - 184815\gamma - 0{,}404\gamma'$, and $\mathrm{ver}(h_5/e) = 1 - 214815\gamma - 0{,}0742\gamma'$
47. Here $\mathrm{ver}(h_4 / e) = 1 - 216990\gamma - 0{,}0508\gamma'$, and $\mathrm{ver}(h_5/e) = 1 - 221275\gamma - 0{,}0666\gamma'$
48. Niiniluoto (1987), section 12.5 especially p.426.
49. One consequence of the naive or content rule is that it does not decide between theories in the following specific situation. Suppose two theories explain the strongest law and respect all instantial data. In that circumstance, Popper would prefer the logically stronger theory, but the success rule 4.2 (p. 132) is indifferent. If it preferred the strongest theory, then it would cease to be functional for approaching the truth. Extension of *R* could put the stronger theory at a disadvantage.
50. Niiniluoto (1987) p.269.
51. *Ibid.* chap. 12.5.
52. See also Bonilla (1996) and Kieseppä (1996).
53. See also Kuipers (1987).
54. Kuipers (1992), p.326.
55. Since 1995, Kuipers presents his rule of theory choice in combination with the HD-method. He formulates his rule of success conditionally: “on the basis of comparative HD-testing, it appears that the theory *Y* will remain more successful than *X*”.
56. See e.g. Niiniluoto (1987), p.380-382.
57. From a private letter to Theo Kuipers dating from 1983.

58. See van Benthem (1996).
59. Preferential reasoning started with Shoham (1988); for conditional logic see Friedman and Halpern (1994). Gärdenfors and Rott (1995) provide an introduction on belief revision.

CHAPTER 5

1. The present chapter is a (heavily) revised version of Zwart (1995).
2. See volume twenty-five of the *BJPS.*
3. Miller (1974).
4. Niiniluoto (1987), p. 459.
5. See Miller (1974), p.175 ff. and Tichý (1974), p.159.
6. Miller (1974), p.176. The gerund 'counting' does not refer to the number of zeroes and ones in general. It refers to counting the number of disagreements between the literals. Moreover, by criticising the "counting method", Miller also attacks his own favoured distance too; the next chapter shows that the resulting distance equals the symmetric difference in the Boolean set algebra of the set of all atomic propositions—the constituent algebra of Chapter 6.
7. Miller (1974), p.176.
8. $X \equiv X'$ means: for all valuations the truth-value of X is the same as that of X' under the corresponding valuation.
9. It has to be noted that the same kind of preference reversal already occurs in the two-proposition case. If voc($\mathscr{L}$)= $\{a, b\}$ and voc($\mathscr{L}'$) = $\{p, q\}$ and $p := a$ and $q := a \leftrightarrow b$ then if $T := a \wedge b,\ Y := \neg a \wedge b,\ X := \neg a \wedge \neg b$, then $Y \leq_T X$, i.e. Y is closer to the truth then X; but after translation $T' := p \wedge q,\ Y' := \neg p \wedge \neg q,\ X := \neg p \wedge q$ and $X' \leq_T Y'$, i.e. X' is closer to the truth then Y'.
10. Miller (1974), p. 176.
11. Miller (1976), p.364-365.
12. This characteristic of Popper's (and Miller's) truth makes Rosenkrantz (1975, endnote 2) complain that "They [*Popper and Miller: SZ*] speak as though the class of all true statements about the world were an unproblematic totality."
13. Urbach (1983), p. 271.
14. Barnes (1991), p.309, the abstract.
15. *Ibid.* p.176.
16. The term is from Barnes (1991, p. 312).
17. The privileged language argument is similar to Goodman's pragmatic solution to his grue-bleen paradox.
18. See Mormann (1988), p. 516-517.
19. Brink and Heidema (1987), p. 547.

20. From a reductionist point of view one could argue that r and w are more fundamental than m and a, since rain and temperature are ontologically distinguishable, substances versus average kinetic energy. Consequently, $\mathscr{L}$ would be more fundamental than $\mathscr{L}'$. We do not indulge in such a metaphysical argument. It is quantum mechanics and the theory of relativity that blurred the so-called ontological distinction between substance and energy, and we should avoid arguments based on ontological categories.
21. Tichý (1978, p.193) and Oddie (1986, p.141) call this the "non-identity" thesis. Veikko Rantala also denies the identity of meanings of *MP* correlated sentences of $\mathscr{L}$ and $\mathscr{L}'$ (*private correspondence*).
22. Oddie (1986, p.67) quotes Carnap (1971]).
23. Tichý (1978), p.192.
24. Miller (1978) extends his argument to first order languages; see also Pearce (1983).
25. Niiniluoto (1987), p. 459, *his italics*.
26. The present section is the result of Miller's question to find out whether the "language dependency" argument also affected the refined structuralist proposals.
27. Miller's verisimilitude definition is invariant to extensional substitutions.
28. The terminology stems from Pearce (1987).
29. Pearce and Rantala (1984), p. 51.
30. See, e.g., Pearce and Rantala (1984).
31. The deduction can be found in the Appendix.
32. See section V.2.1
33. In the context of the Miller example, the shift form a propositional language to a monadic one is of minor importance since the hot, rainy, and windy are also properties of points in time and space.
34. Miller probably denies the difference of the cognitive problems. Regarding a different example with the same form, he claimed: " ... if you relativize the problems to fundamental properties, ... then you get different problems. But these problems are not translations of each other." (*private correspondence*).
35. Goodman (1970), p.27.
36. Lewis (1973), p.91.
37. Niiniluoto (1987) p. 459.
38. *Ibid.* p. 129.
39. Niiniluoto (1998, p.16) seems to accept my analysis.
40. Niiniluoto (1987), p.459.
41. Miller (1978), p. 431.
42. See Miller (1975) and (1978).
43. See e.g. Levinson (1983).

CHAPTER 6

1. Interestingly, in private conversation, Niiniluoto contended that his two dimensional truthlikeness continuum does not subsume my refined verisimilitude proposal.
2. To be precise: $\leq^{\Delta}_{\precsim} \Rightarrow \leq^{+}_{\precsim}$ abbreviates: $\forall\varphi, \psi$: if $\psi \leq^{\Delta}_{\precsim} \varphi$, then $\psi \leq^{+}_{\precsim} \varphi$
3. E.g. Chang and Keisler (1973), p.46-47.
4. Although $\varphi, \psi \in \mathrm{Sent}(\mathcal{L})/\leftrightarrow$ are equivalence classes of sentences in the sequel, I will use them as if they are a single sentence and write $\varphi \models \psi$
5. Clearly, as the constituent and Lindenbaum algebra both are Boolean algebras, the more abstract version of our proposals only uses set algebras; it disregards the identity of the atomic propositions.
6. g is an isomorphism if, in addition to g, g^{-1} is also a homomorphism.
7. $\mathrm{voc}(\mathcal{L}') := \{p \in \mathrm{voc}(\mathcal{L}) \mid \tau \models p\} \cup \{q \notin \mathrm{voc}(\mathcal{L}) \mid q \equiv \neg p, \tau \models \neg p \text{ and } p \in \mathrm{voc}(\mathcal{L})\}$ defines the vocabulary of this new $\mathcal{L}'$. This reformulation does not entail a change in the cognitive problem like the extensional substitutions of Chapter 5.
8. See Davey and Priestley (1990), p. 25.
9. In case the language is finite and the truth τ is complete, the truth ideal consists of the contradiction and τ. The truth ideal is less trivial if τ is incomplete or unaxiomatizable.
10. See e.g. Davey and Priestley (1990), p.145.
11. For more details see Davey and Priestley (1990), p.145).
12. See e.g. Davey and Priestley (1990), p.18-19.
13. See e.g. Stoll (1961), p.263.
14. Intuitively, the verisimilitude of a theory is the similarity between that theory and the true theory regarding *empirical* content. Consequently, it make no sense to contribute verisimilitude to the tautology and the contradiction, which lack empirical content.
15. Counterbalancing logical strength and false consequences is less counter intuitive if the truth is complete; it might even be helpful in language dynamics, where the choice of conceptual frameworks is at stake.
16. This might be the beginning of a compound definition if the truth is incomplete.
17. According to this definition $(p \vee q) \wedge r$ has the same d-n form as $(r \vee q) \wedge p$ since $(p \vee q) \wedge r$ has the same form as $(x \vee y) \wedge z$ which has the same form as $(r \vee q) \wedge p$.
18. The $\precsim_h$-ordering takes the middle course between "best-case" and "worst-case" proposals. A *lexicographical (dictionary) ordering is a best case proposal.* It reads: Let $\alpha_{1,2}, \beta_{1,2} \in \mathrm{AtProp}(\mathcal{L})$, then $\beta_1 \vee \beta_2 \leq \alpha_1 \vee \alpha_2$ iff $\beta_1 < \alpha_1$ or ($\beta_1 \sim \alpha_1$ and $\beta_2 \leq \alpha_2$); (larger disjunctions by iteration; see Davey and Priestley (1990,

p.19)). The *reverse lexicographical* ordering is a worst case proposal. $\beta_1 \vee \beta_2 \leq \alpha_1 \vee \alpha_2$ iff $\beta_2 < \alpha_2$ or ($\beta_2 \sim \alpha_2$ and $\beta_1 \leq \alpha_1$); According to the first $p \leftrightarrow q < p \leftrightarrow \neg q$ and according to the second $p \leftrightarrow \neg q < p \leftrightarrow q$; our $\precsim_h$-ordering does not compare $p \leftrightarrow \neg q$ and $p \leftrightarrow q$. According to the quantitative ordering of Section 6.5 $p \leftrightarrow \neg q$ and $p \leftrightarrow q$ are equally verisimilar. (*This note is the result from remarks made by Heidema and van Benthem).*

19. "A relation R is a *weak ordering of the set A* iff R is transitive and stongly connected in A."

20. See e.g. Suppes (1960), p.84.

21. Niiniluoto (1987), p 192.

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